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See McNeil et al. (2005) for a proof of the three last results.

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Archimedian copulas (contd.)

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Let $\phi(t) = 1 - t$, $t \in [0,1]$. Then $\phi^{[-1]}(t) = \max\{1-t, 0\}$ and $C_\phi(u_1, u_2) := \phi^{[-1]}(\phi(u_1) + \phi(u_2)) = \max\{u_1 + u_2 - 1, 0\} = W(u_1, u_2)$. Thus the Fréchet lower bound is an Archimedian copula.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and an Archimedian copula C generated by ϕ. Then $\rho_\tau(X_1, X_2) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt$ holds.

See Nelsen 1999 for a proof.

Example Kendalls Tau for the Gumbel copula and the Clayton copula

Gumbel: $\phi(t) = (\ln t)^\theta$, $\theta \geq 1$.
$\rho_\tau(\theta) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt = 1 - \frac{1}{\theta}$.

Clayton: $\phi(t) = (t^{-\theta} - 1)/\theta$, $\theta > 0$.
$\rho_\tau(\theta) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)}dt = \frac{\theta}{\theta+2}$.

Multivariate Archimedean copulas

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Definition: A function $g\colon [0,\infty) \to [0,\infty)$ is called completely monotone iff all higher order derivatives of g exist and the following inequalities hold for $k \in \mathbb{N}_*$: $(-1)^k \left(\frac{d^k}{ds^k} g(s)\right)\Big|_{s=t} \geq 0$, $\forall t \in (0,\infty)$.

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Theorem: (Kimberling 1974)
Let $\phi\colon [0,1] \to [0,\infty]$ be a continuous, strictly monotone decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$. The function $C\colon [0,1]^d \to [0,1]$, $C(u) := \phi^{-1}(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_d))$ is a copula for all $d \geq 2$ iff ϕ^{-1} is completely monotone on $[0,\infty)$.

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Lemma: A function $\psi\colon [0,\infty) \to [0,\infty)$ is completely monotone with $\psi(0) = 1$ iff ψ is the Laplace-Stieltjes transform of some distribution function G on $[0,\infty)$, i.e. $\psi(s) = \int_0^\infty e^{-sx} dG(x)$, $s \geq 0$.

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See McNeil et al. (2005) for a proof.

Multivariate Archimedian copulas (contd.)

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Then

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and the distribution function of $U=(U_1,U_2,\ldots,U_d)$ is an Archimedian copula with generator ψ^{-1}.

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- can model a broader class of dependencies
- have a closed form representation
- depend on a small number of parameters in general
- the generator function needs to fulfill quite restrictive technical assumptions

Simulation of Gaussian copulas

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Observe: Consider a symmetric positive definite matrix $R \in \mathbb{R}^{d \times d}$ and its Cholesky factorization $AA^T = R$ with $A \in \mathbb{R}^{d \times d}$. If $Z_1, Z_2, \ldots, Z_d \sim N(0,1)$ are independent, then $\mu + AZ \sim N_d(\mu, R)$.

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Algorithm: for the generation of a random vector $U = (U_1, U_2, \ldots, U_d)$ whose distribution function is the copula C_R^{Ga}, R positive definite with all ones on the main diagonal.

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- Output $U = (U_1, U_2, \ldots, U_d)$; U has distribution function C_R^{Ga}.

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- Output $U = (U_1, U_2, \ldots, U_d)$; $U = (U_1, U_2, \ldots, U_d)$ has distribution function $C^t_{\nu,R}$.

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For $X \sim Gamma(1/\theta, 1)$ with d.f. $f_X(x) = \left(x^{1/\theta - 1} e^{-x}\right) / \Gamma(1/\theta)$ we have: $E(e^{-sX}) = \int_0^\infty e^{-sx} \frac{1}{\Gamma(1/\theta)} x^{1/\theta - 1} e^{-x} dx = (s+1)^{-1/\theta} = \tilde{\varphi}^{-1}(s)$.

Simulation of the Clayton copula ($\theta > 0$)

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- Simulate $X \sim Gamma(1/\theta, 1)$.

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For $\alpha \neq 1$ we get: $X = \delta + \gamma Z \sim St(\alpha, \beta, \gamma, \delta)$.

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Question 2: What are the parameters of the prespecified family of copulas used for the modelling?

Parameter estimation for C_R^{Ga}, $C_{\nu,R}^t$, C_θ^{Cl} and C_θ^{Gu}

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- Compute the spectral decomposition $\hat{R} = \Gamma \Lambda \Gamma^T$ of $\hat{R}$, where Λ is a diagonal matrix, containing the eigenvalues of $\hat{R}$ on the diagonal, and Γ is an orthogonal matrix with the eigenvectors of $\hat{R}$ in its columns.

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- Set $R^* := D\tilde{R}D$ where D is a diagonal matrix with $D_{k,k} = 1/\sqrt{\tilde{R}_{k,k}}$.

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- a non-parametric estimation method;
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where $g_{\xi,R}$ is the cumulative density function of a d-dimensional t-distribution with expectation 0 ξ degrees of freedom and correlation matrix R, and g_ξ is the density function of a univariate standard t-distribution with ξ degrees of freedom.