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For $X \sim Gamma(1/\theta, 1)$ with d.f. $f_X(x) = \left(x^{1/\theta - 1} e^{-x}\right) / \Gamma(1/\theta)$ we have: $E(e^{-sX}) = \int_0^\infty e^{-sx} \frac{1}{\Gamma(1/\theta)} x^{1/\theta - 1} e^{-x} dx = (s+1)^{-1/\theta} = \tilde{\varphi}^{-1}(s)$.

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For $\alpha \neq 1$ we get: $X = \delta + \gamma Z \sim St(\alpha, \beta, \gamma, \delta)$.

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Question 2: What are the parameters of the prespecified family of copulas used for the modelling?

Parameter estimation for C_R^{Ga}, $C_{\nu,R}^t$, C_θ^{Cl} and C_θ^{Gu}

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where

$$\begin{aligned}
(\rho_\tau)_{ij} &= \rho_\tau(X_i, X_j) \\
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Standard empirical estimator of Kendalls Tau:
$\widehat{\rho_\tau}_{ij} = \binom{n}{2}^{-1} \sum_{1 \leq k < l \leq n} sign((X_{k,i} - X_{l,i})(X_{k,j} - X_{l,j}))$.

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Eigenvalue approach (Rousseeuw and Molenberghs 1993)

- Compute the spectral decomposition $\hat{R} = \Gamma\Lambda\Gamma^T$ of $\hat{R}$, where Λ is a diagonal matrix, containing the eigenvalues of $\hat{R}$ on the diagonal, and Γ is an orthogonal matrix with the eigenvectors of $\hat{R}$ in its columns.

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- Set $R^* := D \tilde{R} D$ where D is a diagonal matrix with $D_{k,k} = 1/\sqrt{\tilde{R}_{k,k}}$.

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for $k = 1, 2, \ldots, n$ (see Genest und Rivest 1993).

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- a non-parametric estimation method;
  $\hat{F}_i$ is the empirical distribution function $\hat{F}_i(x) = \frac{1}{n+1} \sum_{t=1}^{n} I_{\{X_{t,i} \leq x\}}$, $1 \leq i \leq d$.

Estimation of the number of the degrees of freedom ν for t-copulas (contd.)

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Estimation of the number of the degrees of freedom ν for t-copulas (contd.)

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$$L(\xi; \hat{U}_1, \hat{U}_2, \ldots, \hat{U}_n) = \Pi_{k=1}^{n} c_{\xi,R}^{t}(\hat{U}_k)$$

and $c_{\xi,R}^{t}$ is the density of the t-copula $C_{\xi,R}^{t}$.

This implies

$$\ln L(\xi; \hat{U}_1, \hat{U}_2, \ldots, \hat{U}_n) =$$

$$\sum_{k=1}^{n} \ln g_{\xi,R}(t_{\xi}^{-1}(\hat{U}_{k,1}), \ldots, t_{\xi}^{-1}(\hat{U}_{k,d})) - \sum_{k=1}^{n}\sum_{j=1}^{d} \ln g_{\xi}(t_{\xi}^{-1}(\hat{U}_{k,j})),$$

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This implies

$$\ln L(\xi; \hat{U}_1, \hat{U}_2, \ldots, \hat{U}_n) =$$

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where

$g_{\xi,R}$ is the joint distribution function of a d-dimensional t-distribution with expectation 0, ξ degrees of freedom and correlation matrix R,

t_ξ is the distribution function of a univariate standard t-distribution with ξ degrees of freedom and g_ξ is its density.