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The standard MC estimator is:

$$\widehat{CVaR}_\alpha^{(MC)}(L) = \frac{1}{\sum_{i=1}^{n} I_{(q_\alpha,+\infty)}(L^{(i)})} \sum_{i=1}^{n} L^{(i)} I_{(q_\alpha,+\infty)}(L^{(i)}),$$

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where $L^{(i)}$ is the value of the loss in the i-th simulation run.
$\widehat{CVaR}_\alpha^{(MC)}(L)$ is unstable, i.e. it has a very high variance, if the number of simulation runs is not very high.

Basics of importance sampling

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Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, P)$ with absolutely continuous distribution function and density function f.

Goal: Determine $\theta = E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x)dx$ for some given function h.

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(1) Simulate X_1,X_2,..., X_n independently with density f.

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In case of rare events, e.g. $h(x) = I_A(x)$ with $P(A) << 1$, the convergence is very slow.

Importance sampling (contd.)

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Let g be a probability density function, such that $f(x) > 0 \Rightarrow g(x) > 0$.

We define the *likelihood ratio* as: $r(x) := \begin{cases} \frac{f(x)}{g(x)} & g(x) > 0 \\ 0 & g(x) = 0 \end{cases}$

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Goal: choose an IS density g such that the variance of the IS estimator is much smaller than the variance of the standard MC-estimator.

$$var\left(\hat{\theta}_n^{(IS)}\right) = \frac{1}{n^2}(E_g(h^2(X)r^2(X)) - \theta^2)$$

$$var\left(\hat{\theta}_n^{(MC)}\right) = \frac{1}{n^2}(E(h^2(X)) - \theta^2)$$

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Goal: choose g such that $E_g(h^2(X)r^2(X))$ becomes small, i.e. such that $r(x)$ is small for $x \geq c$. Equivalently, the event $X \geq c$ should be more probable under density g than under density f.

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For example for the estimation of the tail probability?

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(A unique solution of the above equality exists for all relevant values of c, see e.g. Embrechts et al. for a proof).

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We have $\frac{dP}{dQ_t} = M_X(t)\exp(-tX) =: r_t(X)$.
The IS algorithm does not change: Simulate independent realisations of X_i in $(\Omega, \mathcal{F}, Q_t)$ and set $\hat{\theta}_n^{(IS)} = (1/n)\sum_{i=1}^n x_i r_t(x_i)$, where x_i is the realizations of X_i, for $i \in \overline{1, n}$.

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Let Z be a vector of economical impact factors, such that $Y_i | Z$ are independent and $Y_i | (Z = z) \sim Bernoulli(p_i(z))$, $\forall i = 1, 2, \ldots, m$.

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Goal: Estimation of $\theta = P(L \geq c)$ by means of IS, for some given c with $c >> E(L)$.

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Simplified case: Y_i are independent for $i = 1, 2, \ldots, m$.
Let $\Omega = \{0, 1\}^m$ be the state space of the random vector Y.
Consider the probability measure P in Ω:

$$\mathbb{P}(\{y\}) = \prod_{i=1}^{m} \mathbb{P}(Y_i = y_i) = \prod_{i=1}^{m} \bar{p}_i^{y_i}(1 - \bar{p}_i)^{1-y_i}, \text{for all } y \in \{0, 1\}^m.$$

The moment generating function of L is
$M_L(t) = E(e^{tL}) = \prod_{i=1}^{m}(e^{te_i}\bar{p}_i + 1 - \bar{p}_i)$.

IS in the case of Bernoulli mixture models (contd.)

Consider a probability measure Q_t:

$$Q_t(\{y\}) = \prod_{i=1}^{n} \left(\frac{\exp\{te_i y_i\}}{\exp\{te_i\}\bar{p}_i + 1 - \bar{p}_i} \bar{p}_i^{y_i} (1 - \bar{p}_i)^{1-y_i} \right).$$

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Let $\bar{q}_{t,i}$ be new default probabilities

$$\bar{q}_{t,i} := \exp\{te_i\}\bar{p}_i / (\exp\{te_i\}\bar{p}_i + 1 - \bar{p}_i).$$

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Thus after applying the exponential tilting the default indicators are independent with new default probabilities $\bar{q}_{t,i}$.

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$\lim_{t\to\infty} \bar{q}_{t,i} = 1$ and $\lim_{t\to-\infty} \bar{q}_{t,i} = 0$ imply that $E^{Q_t}(L)$ takes all values in $(0, \sum_{i=1}^{m} e_i)$ for $t \in \mathbb{R}$.

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Choose t, such that $\sum_{i=1}^{m} e_i \bar{q}_{t,i} = c$.