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Algorithm: IS for the conditional loss distribution

(1) For a given z proceed as in the simplified case; use the conditional default probabilities $p_i(z)$ and solve the equation

$$\sum_{i=1}^{m} e_i \frac{\exp\{te_i\} p_i(z)}{\exp\{te_i\} p_i(z) + 1 - p_i(z)} = c\,.$$

The solution $t = t(c, z)$ specifies the correct *degree of tilting*.

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(2) Generate n_1 conditional realisations of the vector of default indicators $(Y_1, \ldots, Y_m)$, Y_i are simulated from $Bernoulli(q_i)$, $i = 1, 2, \ldots, m$, with

$$q_i = \frac{\exp\{t(c, z)e_i\} p_i(z)}{\exp\{t(c, z)e_i\} p_i(z) + 1 - p_i(z)}\,.$$

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(3) Let $M_L(t,z) := \prod[\exp\{t(c,z)e_i\}p_i(z) + 1 - p_i(z)]$ be the conditional moment generating function of L. Let $L^{(1)}$, $L^{(2)}$,...,$L^{(n_1)}$ be the n_1 conditional realisations of L for the n_1 simulated realisations of $Y_1, Y_2, \ldots, Y_m$. Compute the *IS*-estimator for the tail probability of the conditional loss distribution:

$$\hat{\theta}_{n_1}^{(IS)}(z) = M_L(t(c,z),z)\frac{1}{n_1}\sum_{j=1}^{n_1} I_{L^{(j)}\geq c}\exp\{-t(c,z)L^{(j)}\}L^{(j)}.$$

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Naive approach: Generate many realisations z of the impact factors Z and compute $\hat{\theta}_{n_1}^{(IS)}(z)$ for every one of them. The required estimator is the average of $\hat{\theta}_{n_1}^{(IS)}(z)$ over all realisations z.
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A better alternative: IS for the impact factors.

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(3) compute the IS estimator for the independent excess probability:

$$\hat{\theta}_n^{(IS)} = \frac{1}{n}\sum_{i=1}^{n} r_\mu(z_i)\hat{\theta}_{n_1}^{(IS)}(z_i)$$

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See Glasserman und Li (2003) for some numerical solution approaches.