Risk and Management: Goals and Perspective

Etymology: Risicare

Risk (Oxford English Dictionary): (Exposure to) the possibility of loss, injury, or other adverse or unwelcome circumstance; a chance or situation involving such a possibility.
Finance: The possibility that an actual return on an investment will be lower than the expected return.

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Risk management: is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events or to maximize the realization of opportunities.
Risk management's objective is to assure uncertainty does not deflect the endeavor from the business goals.

Risk and Management: Goals and Perspective

Subject of risk managment:

- Identification of risk sources (determination of exposure)
- Assessment of risk dependencies
- Measurement of risk
- Handling with risk
- Control and supervision of risk
- Monitoring and early detection of risk
- Development of a well structured risk management system

Risk and Management: Goals and Perspective

Main questions addressed by strategic risk managment:

- Which are the strategic risks?
- Which risks should be carried by the company?
- Which instruments should be used to control risk?
- What resources are needed to cover for risk?
- What are the risk adjusted measures of success used as steering mechanisms?

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Examples: standard deviation, quantile of the loss distribution, ...

Types of risk

For an organization risk arises through events or activities which could prevent the organization from fulfilling its goals and executing its strategies.

Financial risk:

- Market risk
- Credit risk
- Operational risk
- Liquidity risk, legal (judicial) risk, reputational risk

The goal is to estimate these risks as precisely as possible, ideally based on the loss distribution (LD).

Regulation and supervision

1974: Establishment of Basel Committee on Banking Supervision (BCBS).

- **Risk capital** depending on GD/LD.
- Suggestions and guidelines on the requirements and methods used to **compute the risk capital**. Aims at **internationally accepted standards** for the computation of the risk capital and **statutory dispositions** based on those standards.
- **Control** by the supervision agency.

Regulation and supervision: Historical view

1988 Basel I: International minimum capital requirements especially with respect to (w.r.t.) credit risk.

1996 Standardised models for the assessment of market risk with an option to use value at risk (VaR) models in larger banks

2007 Basel II: minimum capital requirements w.r.t. credit risk, market risk and operational risk, procedure of control by supervision agencies, market discipline (see `http://www.bis.org`).

2010 BASEL III - Improvement and further development of BASEL II w.r.t. applicability, operational risk und liquidity risk

2017 BASEL IV - new standards for the computation of the minimum capital requirements including a standartized lower bound for risk-weighted assets

Assessment of the loss function

Loss operators
$V(t)$ - Value of portfolio at time t
Time unit Δt
Loss in time interval $[t, t + \Delta t]$: $L_{[t,t+\Delta t]} := -(V(t + \Delta t) - V(t))$
Discretisation of time: $t_n := n\Delta t$, $n = 0, 1, 2, \ldots$

$$L_{n+1} := L_{[t_n, t_{n+1}]} = -(V_{n+1} - V_n), \text{ where } V_n := V(n\Delta t)$$

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The portfolio consists of α_i units of asset A_i with price $S_{n,i}$ at time t_n, $i = 1, 2, \ldots, d$.

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Let $Z_{n,i} := \ln S_{n,i}$, $X_{n+1,i} := \ln S_{n+1,i} - \ln S_{n,i}$
Let $w_{n,i} := \alpha_i S_{n,i}/V_n$, $i = 1, 2, \ldots, d$, be the relative portfolio weights.

Loss operator of an asset portfolio (cont.)

The following holds:

$$L_{n+1} := -\sum_{i=1}^{d} \alpha_i S_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =$$

$$-V_n \sum_{i=1}^{d} w_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =: l_n(X_{n+1})$$

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Linearisation $e^x = 1 + x + o(x^2) \sim 1 + x$ implies

$$L^{\Delta}_{n+1} = -V_n \sum_{i=1}^{d} w_{n,i} X_{n+1,i} =: l^{\Delta}_n(X_{n+1}),$$

where L_{n+1} ($L^{\Delta}_{n+1]}$) is the (linearised) loss function and l_n (l^{Δ}_n) is the (linearised) loss operator.

The general case

Let $V_n = f(t_n, Z_n)$ and $Z_n = (Z_{n,1}, \ldots, Z_{n,d})$, where Z_n is a vector of risk factors

Risk factor changes: $X_{n+1} := Z_{n+1} - Z_n$

$L_{n+1} = -\left(f(t_{n+1}, Z_n + X_{n+1}) - f(t_n, Z_n) \right) =: l_n(X_{n+1})$, where

$l_n(x) := -\left(f(t_{n+1}, Z_n + x) - f(t_n, Z_n) \right)$ is the loss operator.

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The linearised loss:

$$L^{\Delta}_{n+1} = -\left(f_t(t_n, Z_n)\Delta t + \sum_{i=1}^{d} f_{z_i}(t_n, Z_n) X_{n+1,i} \right),$$

where f_t and f_{z_i} are the partial derivatives of f.

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A currency forward or an FX forward (FXF) is a contract between two parties to buy/sell an amount $\bar{V}$ of a foreign currency at a future time T for a specified exchange rate $\bar{e}$. The party who is going to buy the foreign currency is said to hold a long position and the party who will sell holds a short position.

Example 1:

A bond portfolio
Let $B(t, T)$ be the price of the ZCB with maturity T at time $t < T$.

The **continuously compounded yield**, $y(t, T) := -\frac{1}{T-t} \ln B(t, T)$, represents the continuous interest rate which would have been dealt with at time t as being constant for the whole interval $[t, T]$.

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Portfolio value at time t_n:

$$V_n = \sum_{i=1}^{d} \alpha_i B(t_n, T_i) = \sum_{i=1}^{d} \alpha_i exp\{-(T_i - t_n) Z_{n,i}\} = f(t_n, Z_n),$$

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Let $X_{n+1,i} := Z_{n+1,i} - Z_{n,i}$ be the risk factor changes.

A bond portfolio (contd.)

$$l_{[n]}(x) = -\sum_{i=1}^{d} \alpha_i B(t_n, T_i) \left(exp\{Z_{n,i}\Delta t - (T_i - t_{n+1})x_i\} - 1 \right)$$

$$L^{\Delta}_{n+1} = -\sum_{i=1}^{d} \alpha_i B(t_n, T_i) \left(Z_{n,i}\Delta t - (T_i - t_{n+1})X_{n+1,i} \right)$$

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The party who buys the foreign currency holds a **long position**. The party who sells holds a **short position**.

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Observation:
A long position over $\bar{V}$ units of an FX forward with maturity T
is equivalent to
(1) a long position over $\bar{V}$ units of a foreign zero-coupon bond (ZCB) with maturity T and (2) a short position over $\bar{e}\bar{V}$ units of a domestic zero-coupon bond with maturity T.

A currency forward portfolio (contd.)

Assumptions:
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$V_T = \bar{V}(e(T) - \bar{e})$.

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The linearized loss: $L^{\Delta}_{n+1} = -V_n(Z_{n,2}\Delta t + X_{n+1,1} - (T - t_{n+1})X_{n+1,2})$
with $X_{n+1,1} := \ln e(t_{n+1}) - \ln e(t_n)$ and $X_{n+1,2} := y^f(t_{n+1}, T) - y^f(t_n, T)$

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The greeks: C_t - theta, C_S - delta, C_r - rho, C_σ - Vega

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- Notational amount: weighted sum of notational values of individual securities weighted by a prespecified factor for each asset class

  e.g. in Basel I (1998):

  $$\textbf{Cooke Ratio} = \frac{\text{regulatory capital}}{\text{risk-weighted sum}} \geq 8\%$$

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  Disadvantages: no difference between long and short positions, diversification effects are not considered

- **Coefficients of sensitivity** with respect to risk factors

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  Z_n ist a vector of d risk factors

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  Portfolio risk:

  $\Psi[\chi, w] = \max\{w_1 l_{[n]}(X_1), w_2 l_{[n]}(X_2), \ldots, w_N l_{[n]}(X_N)\}$

Example: SPAN rules applied at CME (see Artzner et al., 1999)
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Scenarios 1 to 8		Scenarios 9 to 14	
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Scenarios i, $i = 15, 16$ represent an extreme increase or decrease of the future price, respectively. The weights are $w_i = 1$, for $i \in \{1, 2, \ldots, 14\}$, and $w_i = 0.35$, for $i \in \{15, 16\}$.

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An appropriate model (zB. Black-Scholes) is used to generate the option prices in the different scenarios.

Risk measures based on the loss distribution

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1. **The standard deviation** $std(L) := \sqrt{\sigma^2(F_L)}$

   It is used frequently in portfolio theory.

   Disadvantages:

   - STD exists only for distributions with $E(F_L^2) < \infty$, not applicable to leptocurtic ("fat tailed") loss distributions;
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Example

$L_1 \sim N(0,2)$, $L_2 \sim t_4$ (Student's t-distribution with $m = 4$ degrees of freedom)
$\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold
However the probability of losses is much larger for L_2 than for L_1.
Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$!

2. **Value at Risk** ($VaR_\alpha(L)$)

   Let L be the loss distribution with distribution function F_L and let $\alpha \in (0,1)$ be a given confindence level.

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Definition: Let $F \colon A \to B$ be an increasing function. The function $F^{\leftarrow} \colon B \to A \cup \{-\infty, +\infty\}, y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \geq y\}$ is called *generalized inverse function* of F.

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If F is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

Exercise: Compute $F^{\leftarrow}$ for $F\colon [0, +\infty) \to [0,1]$ with

$$F(x) = \begin{cases} 1/2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Value at Risk (contd.)

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a (monotone increasing) distribution function and $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ be α-quantile of F.

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Example: Let $L \sim N(\mu, \sigma^2)$. Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma \Phi^{-1}(\alpha)$ holds, where Φ is the d.f. of a r.v. $X \sim N(0,1)$.

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Exercise: Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d., i.e. $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^{\Delta})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^{Δ} is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

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If F_L **is discrete** the *generalized CVaR* is defined as follows:

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Conditional Value at Risk (contd.)
Example 1:

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

(b) Let the distribution function F_L of the loss function L be given as follows : $F_L(x) = 1 - (1+\gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0,1)$. Compute $CVaR_\alpha(L)$.

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Let $L \sim N(0, 1)$. Let ϕ und Φ be the density and the distribution function of L, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

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Exercise:

Let the loss L be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of L is

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu+(t_\nu^{-1}(a))^2}{\nu-1}\right)$, where t_ν is the distribution function of L.

Methods for the computation of VaR und CVaR

Consider the portfolio value $V_m = f(t_m, Z_m)$, where Z_m is the vector of risk factors.

Let the loss function over the interval $[t_m, t_{m+1}]$ be given as $L_{m+1} = l_{[m]}(X_{m+1})$, where X_{m+1} is the vector of the risk factor changes, i.e.

$$X_{m+1} = Z_{m+1} - Z_m.$$

Consider observations (historical data) of risk factor values $Z_{m-n+1}, \ldots, Z_m$.
How to use these data to compute/estimate $VaR(L_{m+1})$, $CVaR(L_{m+1})$?

The empirical VaR and the empirical CVaR

Let $x_1, x_2, \ldots, x_n$ be a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function F.

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Assumption: $x_1 > x_2 > \ldots > x_n$. Then $q_\alpha(F_n) = x_{[n(1-\alpha)]+1}$ holds, where $[y] := \sup\{n \in \mathbb{N} \colon n \leq y\}$ for every $y \in \mathbb{R}$.

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Lemma

Let $\hat{q}_\alpha(F) := q_\alpha(F_n)$ and let F be a strictly increasing function. Then $\lim_{n\to\infty} \hat{q}_\alpha(F) = q_\alpha(F)$ holds $\forall \alpha \in (0,1)$, i.e. the estimator $\hat{q}_\alpha(F)$ is consistent.

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The empirical estimator of CVaR is $\widehat{\text{CVaR}}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_k}{[(n(1-\alpha)]+1}$

A non-parametric bootstrapping approach to compute the confidence interval for the estimator

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Goal: computation of an estimator of a certain parameter θ depending on F, e.g. $\theta = q_\alpha(F)$, and the corresponding confidence interval.

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Let $\hat{\theta}(x_1, \ldots, x_n)$ be an estimator of θ, e.g. $\hat{\theta}(x_1, \ldots, x_n) = x_{[(n(1-\alpha)]+1}$
$\theta = q_\alpha(F)$.
The required confidence interval is an (a, b) with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level p.

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Case I: F is known.
Generate N samples $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}$, $1 \le i \le N$, by simulation from F (N should be large)
Let $\tilde{\theta}_i = \hat{\theta}\left(\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}\right)$, $1 \le i \le N$.

Case I (cont.)
The empirical distribution function of $\hat{\theta}(x_1, x_2, \dots, x_n)$ is given as

$$F_N^{\hat{\theta}} := \frac{1}{N} \sum_{i=1}^{N} I_{[\tilde{\theta}_i, \infty)}$$

and it tends to $F^{\hat{\theta}}$ for $N \to \infty$.

The required conficence interval is given as

$$\left(q_{\frac{1-p}{2}}(F_N^{\hat{\theta}}), q_{\frac{1+p}{2}}(F_N^{\hat{\theta}}) \right)$$

(assuming that the sample sizes N und n are large enough).

Case II: F is not known. Apply bootstrapping!
The empirical distribution function of X_i, $1 \le i \le n$, is given as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i,\infty)}(x).$$

For n large $F_n \approx F$ holds.

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Generate samples from F_n be choosing n elementes in $\{x_1, x_2, \ldots, x_n\}$ and putting every element back to the set immediately after its choice
Assume N such samples are generated: $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \leq i \leq N$.

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.
The empirical distribution of θ_i^* is given as $F_N^{\theta^*}(x) = \frac{1}{N}\sum_{i=1}^{N} I_{[\theta_i^*,\infty)}(x)$;
it approximates the d.f. $F^{\hat{\theta}}$ of $\hat{\theta}(X_1, X_2, \ldots, X_n)$ for $N \to \infty$.

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A confidence interval (a, b) with confidence level p is given by

$$a = q_{(1-p)/2}(F_N^{\theta^*}) \qquad b = q_{(1+p)/2}(F_N^{\theta^*}).$$

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Thus $a = \theta^*_{[N(1+p)/2]+1}$, $b = \theta^*_{[N(1-p)/2]+1}$, where $\theta_1^* \geq \ldots \geq \theta_N^*$.

Summary of the non-parametric bootstrapping approach to compute confidence intervals

Input: Sample $x_1, x_2, \ldots, x_n$ of the i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function F and an estimator $\hat{\theta}(x_1, x_2, \ldots, x_n)$ of an unknown parameter $\theta(F)$, A confidence level $p \in (0, 1)$.

Output: A confidence interval I_p for θ with confidence level p.

- Generate N new Samples $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \leq i \leq N$, by chosing elements in $\{x_1, x_2, \ldots, x_n\}$ and putting them back right after the choice.
- Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.
- Setz $I_p := \left(\theta^*_{[N(1+p)/2]+1,N}, \theta^*_{[N(1-p)/2]+1,N}\right)$, where $\theta^*_{1,N} \geq \theta^*_{2,N} \geq \ldots \theta^*_{N,N}$ is obtained by sorting $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$.

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Input: A sample $x_1, x_2, \ldots, x_n$ of the random variables X_i, $1 \le i \le n$, i.i.d. with unknown continuous distribution function F, a confidence level $p \in (0, 1)$.

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Input: A sample $x_1, x_2, \ldots, x_n$ of the random variables X_i, $1 \leq i \leq n$, i.i.d. with unknown continuous distribution function F, a confidence level $p \in (0, 1)$.

Output: A $p' \in (0, 1)$, with $p \leq p' \leq p + \epsilon$, for some small ϵ, and a confidence interval (a, b) for $q_\alpha(F)$, i.e. $a = a(x_1, x_2, \ldots, x_n)$, $b = b(x_1, x_2, \ldots, x_n)$, such that

$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F) \leq (1-p)/2$ holds.

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$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F) \leq (1-p)/2$ holds.

Assume w.l.o.g. that the sample is sorted $x_1 \geq x_2 \geq \ldots \geq x_n$.

Determine $i > j$, $i, j \in \{1, 2, \ldots, n\}$, and the smallest $p' > p$, such that

$$P\left(x_i < q_\alpha(F) < x_j\right) = p' \qquad (*) \qquad \text{and}$$

$$P\left(x_i \geq q_\alpha(F)\right) \leq (1-p)/2 \text{ and } P\left(x_j \leq q_\alpha(F)\right) \leq (1-p)/2 (**).$$

An approximative solution without bootstrapping (contd.)

Let $Y_\alpha := \#\{x_k : x_k > q_\alpha(F)\}$

An approximative solution without bootstrapping (contd.)

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We get $P(x_j \leq q_\alpha(F)) \approx P(x_j < q_\alpha(F)) = P(Y_\alpha \leq j - 1)$
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Compute $P(x_j \leq q_\alpha(F))$ and $P(x_i \geq q_\alpha(F))$ for different i and j until indices $i, j \in \{1, 2, \ldots, n\}$, $i > j$, which fulfill $(**)$ are found.

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Set $b := x_j$ and $a := x_i$.

Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
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Assumption: the historically realized losses are i.i.d.
The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses to obtain $l_{1,n} \geq l_{2,n} \geq \ldots \geq l_{n,n}$.

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The aggregated loss over a given time interval
For example, for 10 time units, compute $\lfloor n/10 \rfloor$ aggregated loss realizations $l_k^{(10)}$ over the time intervals $[m-n+10(k-1)+1, m-n+10(k-1)+10]$, $k = 1,\ldots,\lfloor n/10 \rfloor$:

$$l_k^{(10)} = l_{[m]}\left(\sum_{j=1}^{10} x_{m-n+10(k-1)+j}\right).$$

Then compute the empirical estimators of the risk measures.

Historical simulation (contd.)

Advantages:

- simple implementation
- considers intrinsically the dependencies between the elements of the vector of the risk factors changes $X_{m-k} = (X_{m-k,1}, \ldots, X_{m-k,d})$.

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- simple implementation
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Disadvantages:

- lots of historical data needed to get good estimators
- the estimated loss cannot be larger than the maximal loss experienced in the past

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$L_{m+1}^{\Delta} = l_m^{\Delta}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
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and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

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Estimator for VaR: $\widehat{VaR}(L_{m+1}) = -Vw^T\hat{\mu} + V\sqrt{w^T\hat{\Sigma}w}\,\phi^{-1}(\alpha)$

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Advantages:

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Disadvantages:

- Linearisation is not always appropriate, only for a short time horizon reasonable
- The normal distribution assumption could lead to underestimation of risks and should be argued upon (e.g. in terms of historical data)

(iii) Monte-Carlo approach

(1) historical observations of risk factor changes $X_{m-n+1}, \ldots, X_m$.

(2) assumption on a parametric model for the cumulative distribution function of X_k, $m-n+1 \le k \le m$;
e.g. a common distribution function F and independence

(3) estimation of the parameters of F.

(4) generation of N samples $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ from F ($N \gg 1$) and computation of the losses $l_k = l_{[m]}(\tilde{x}_k)$, $1 \le k \le N$

(5) computation of the empirical distribution of the loss function L_{m+1}:

$$\hat{F}_N^{L_{m+1}}(x) = \frac{1}{N} \sum_{k=1}^{N} I_{[l_k,\infty)}(x).$$

(5) computation of estimates for the VaR and CVAR of the loss function: $\widehat{VaR}(L_{m+1}) = (\hat{F}_N^{L_{m+1}}\Big) = l_{[N(1-\alpha)]+1,N}$,

$\widehat{CVaR}(L_{m+1}) = \frac{\sum_{k=1}^{[N(1-\alpha)]+1} l_{k,N}}{[N(1-\alpha)]+1}$,

where the losses are sorted as $l_{1,N} \ge l_{2,N} \ge \ldots \ge l_{N.N}$.

Monte-Carlo approach (contd.)

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Disadvantages:

- computationally expensive; a large number of simulations needed to obtain good estimates

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price be S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

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Alternative: Monte-Carlo simulation.

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It is simple to simulate from this model.