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**The greeks**: $C_t$ - theta, $C_S$ - delta, $C_r$ - rho, $C_\sigma$ - Vega

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- Notational amount: weighted sum of notational values of individual securities weighted by a prespecified factor for each asset class

  e.g. in Basel I (1998):

  $$\textbf{Cooke Ratio} = \frac{\text{regulatory capital}}{\text{risk-weighted sum}} \geq 8\%$$

  $$\text{Gewicht} := \begin{cases} 0\% & \text{for claims on governments and supranationals (OECD)} \\ 20\% & \text{claims on banks} \\ 50\% & \text{claims on individual investors with mortgage securities} \\ 100\% & \text{claims on the private sector} \end{cases}$$

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  Disadvantages: no difference between long and short positions, diversification effects are not considered

- **Coefficients of sensitivity** with respect to risk factors

  Portfolio value at time $t_n$: $V_n = f(t_n, Z_n)$,
  $Z_n$ ist a vector of $d$ risk factors

  Sensitivity coefficients: $f_{z_i} = \frac{\delta f}{\delta z_i}(t_n, Z_n)$, $1 \leq i \leq d$

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  Portfolio risk:

  $\Psi[\chi, w] = \max\{w_1 l_{[n]}(X_1), w_2 l_{[n]}(X_2), \ldots, w_N l_{[n]}(X_N)\}$

**Example**: SPAN rules applied at CME (see Artzner et al., 1999)
A portfolio consists of units of a certain future contract and *put* and *call options* on the same contract with the same maturity.

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Scenarios $i$, $1 \leq i \leq 14$:

| Scenarios 1 to 8 | | Scenarios 9 to 14 | |
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| Volatility | Price of the future | Volatility | Price of the future |
| $\nearrow$ | $\nearrow \frac{1}{3} * Range$ | $\nearrow$ | $\searrow \frac{1}{3} * Range$ |
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Scenarios $i$, $i = 15, 16$ represent an extreme increase or decrease of the future price, respectively. The weights are $w_i = 1$, for $i \in \{1, 2, \ldots, 14\}$, and $w_i = 0.35$, for $i \in \{15, 16\}$.

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An appropriate model (zB. Black-Scholes) is used to generate the option prices in the different scenarios.

**Risk measures based on the loss distribution**

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1. **The standard deviation** $std(L) := \sqrt{\sigma^2(F_L)}$

   It is used frequently in portfolio theory.

   Disadvantages:

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**Example**

$L_1 \sim N(0,2)$, $L_2 \sim t_4$ (Student's $t$-distribution with $m = 4$ degrees of freedom)
$\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold
However the probability of losses is much larger for $L_2$ than for $L_1$.
Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$!

2. **Value at Risk** ($VaR_\alpha(L)$)

   Let $L$ be the loss distribution with distribution function $F_L$ and let $\alpha \in (0,1)$ be a given confindence level.

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**Definition:** Let $F \colon A \to B$ be an increasing function. The function $F^{\leftarrow} \colon B \to A \cup \{-\infty, +\infty\}, y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \ge y\}$ is called *generalized inverse function* of $F$.

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If $F$ is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

**Exercise:** Compute $F^{\leftarrow}$ for $F \colon [0, +\infty) \to [0,1]$ with

$$F(x) = \begin{cases} 1/2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

## Value at Risk (contd.)

Let $F\colon \mathbb{R} \to \mathbb{R}$ be a (monotone increasing) distribution function and $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ be $\alpha$-quantile of $F$.

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**Example:** Let $L \sim N(\mu, \sigma^2)$. Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma \Phi^{-1}(\alpha)$ holds, where $\Phi$ is the d.f. of a r.v. $X \sim N(0, 1)$.

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**Exercise:** Consider a portfolio consisting of 5 pieces of an asset $A$. The today's price of $A$ is $S_0 = 100$. The daily logarithmic returns are i.i.d., i.e. $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let $L_1$ be the 1-day portfolio loss in the time interval (today, tomorrow).

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^{\Delta})$, where $L_{100}$ is the 100-day portfolio loss over a horizon of 100 days starting with today. $L_{100}^{\Delta}$ is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

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**If** $F_L$ **is discrete** the *generalized CVaR* is defined as follows:

$$GCVaR_\alpha(L) := \frac{1}{1-\alpha} \left[ E(LI_{[q_\alpha(L),\infty)}) + q_\alpha \left( 1 - \alpha - P(L > q_\alpha(L)) \right) \right]$$

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**Lemma** Let $\alpha$ be a given confidence level and $L$ a continuous loss function with distribution $F_L$.
Then $CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_p(L)dp$ holds.

**Conditional Value at Risk (contd.)**
**Example 1:**

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

(b) Let the distribution function $F_L$ of the loss function $L$ be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0, 1)$. Compute $CVaR_\alpha(L)$.

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(b) Let the distribution function $F_L$ of the loss function $L$ be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0, 1)$. Compute $CVaR_\alpha(L)$.

**Example 2:**

Let $L \sim N(0, 1)$. Let $\phi$ und $\Phi$ be the density and the distribution function of $L$, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

Let $L' \sim N(\mu, \sigma^2)$. Show that $CVaR_\alpha(L') = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

## Conditional Value at Risk (contd.)

**Example 1:**

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

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**Exercise:**

Let the loss $L$ be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of $L$ is

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where $t_\nu$ is the distribution function of $L$.

# Methods for the computation of VaR und CVaR

Consider the portfolio value $V_m = f(t_m, Z_m)$, where $Z_m$ is the vector of risk factors.

Let the loss function over the interval $[t_m, t_{m+1}]$ be given as $L_{m+1} = l_{[m]}(X_{m+1})$, where $X_{m+1}$ is the vector of the risk factor changes, i.e.

$$X_{m+1} = Z_{m+1} - Z_m.$$

Consider observations (historical data) of risk factor values $Z_{m-n+1}, \ldots, Z_m$.
How to use these data to compute/estimate $VaR(L_{m+1})$, $CVaR(L_{m+1})$?

**The empirical VaR and the empirical CVaR**

Let $x_1, x_2, \ldots, x_n$ be a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function $F$.

**The empirical VaR and the empirical CVaR**

Let $x_1, x_2, \ldots, x_n$ be a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function $F$.

The empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{[x_k, +\infty)}(x)$$

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Assumption: $x_1 > x_2 > \ldots > x_n$. Then $q_\alpha(F_n) = x_{[n(1-\alpha)]+1}$ holds, where $[y] := \sup\{n \in \mathbb{N} \colon n \leq y\}$ for every $y \in \mathbb{R}$.

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## Lemma

*Let $\hat{q}_\alpha(F) := q_\alpha(F_n)$ and let $F$ be a strictly increasing function. Then $\lim_{n\to\infty} \hat{q}_\alpha(F) = q_\alpha(F)$ holds $\forall \alpha \in (0,1)$, i.e. the estimator $\hat{q}_\alpha(F)$ is consistent.*

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The empirical estimator of CVaR is $\widehat{\text{CVaR}}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_k}{[(n(1-\alpha)]+1}$

## A non-parametric bootstrapping approach to compute the confidence interval for the estimator

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Let $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution function $F$ and let $x_1 > x_2 > \ldots > x_n$ be an ordered sample of $F$.

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Let $\hat{\theta}(x_1, \ldots, x_n)$ be an estimator of $\theta$, e.g. $\hat{\theta}(x_1, \ldots, x_n) = x_{[(n(1-\alpha)]+1}$
$\theta = q_\alpha(F)$.
The required confidence interval is an $(a, b)$ with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level $p$.

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**Case I**: $F$ is known.
Generate $N$ samples $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}$, $1 \le i \le N$, by simulation from $F$ ($N$ should be large)
Let $\tilde{\theta}_i = \hat{\theta}\left(\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}\right)$, $1 \le i \le N$.

**Case I (cont.)**
The empirical distribution function of $\hat{\theta}(x_1, x_2, \ldots, x_n)$ is given as

$$F_N^{\hat{\theta}} := \frac{1}{N} \sum_{i=1}^{N} I_{[\tilde{\theta}_i, \infty)}$$

and it tends to $F^{\hat{\theta}}$ for $N \to \infty$.

The required conficence interval is given as

$$\left( q_{\frac{1-p}{2}}(F_N^{\hat{\theta}}), q_{\frac{1+p}{2}}(F_N^{\hat{\theta}}) \right)$$

(assuming that the sample sizes $N$ und $n$ are large enough).

**Case II**: F is not known. Apply bootstrapping!
The empirical distribution function of $X_i$, $1 \leq i \leq n$, is given as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i, \infty)}(x).$$

For n large $F_n \approx F$ holds.

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Generate samples from $F_n$ be choosing $n$ elementes in $\{x_1, x_2, \ldots, x_n\}$ and putting every element back to the set immediately after its choice

Assume $N$ such samples are generated: $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \le i \le N$.

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.

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The empirical distribution of $\theta_i^*$ is given as $F_N^{\theta^*}(x) = \frac{1}{N}\sum_{i=1}^{N} I_{[\theta_i^*,\infty)}(x)$;
it approximates the d.f. $F^{\hat{\theta}}$ of $\hat{\theta}(X_1, X_2, \ldots, X_n)$ for $N \to \infty$.

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A confidence interval $(a, b)$ with confidence level $p$ is given by

$$a = q_{(1-p)/2}(F_N^{\theta^*}) \qquad b = q_{(1+p)/2}(F_N^{\theta^*}).$$

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$$a = q_{(1-p)/2}(F_N^{\theta^*}) \qquad b = q_{(1+p)/2}(F_N^{\theta^*}) \,.$$

Thus $a = \theta^*_{[N(1+p)/2]+1}$, $b = \theta^*_{[N(1-p)/2]+1}$, where $\theta_1^* \ge \dots \ge \theta_N^*$.

**Summary of the non-parametric bootstrapping approach to compute confidence intervals**

**Input:** Sample $x_1, x_2, \dots, x_n$ of the i.i.d. random variables $X_1, X_2, \dots, X_n$ with distribution function $F$ and an estimator $\hat{\theta}(x_1, x_2, \dots, x_n)$ of an unknown parameter $\theta(F)$, A confidence level $p \in (0, 1)$.

**Output:** A confidence interval $I_p$ for $\theta$ with confidence level $p$.

- Generate $N$ new Samples $x_1^{*(i)}, x_2^{*(i)}, \dots, x_n^{*(i)}$, $1 \le i \le N$, by chosing elements in $\{x_1, x_2, \dots, x_n\}$ and putting them back right after the choice.
- Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \dots, x_n^{*(i)}\right)$.
- Setz $I_p := \left(\theta^*_{[N(1+p)/2]+1,N}, \theta^*_{[N(1-p)/2]+1,N}\right)$, where $\theta^*_{1,N} \ge \theta^*_{2,N} \ge \dots \theta^*_{N,N}$ is obtained by sorting $\theta_1^*, \theta_2^*, \dots, \theta_N^*$ .

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**Input:** A sample $x_1, x_2, \ldots, x_n$ of the random variables $X_i$, $1 \leq i \leq n$, i.i.d. with unknown continuous distribution function $F$, a confidence level $p \in (0, 1)$.

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**Output:** A $p' \in (0, 1)$, with $p \leq p' \leq p + \epsilon$, for some small $\epsilon$, and a confidence interval $(a, b)$ for $q_\alpha(F)$, i.e. $a = a(x_1, x_2, \ldots, x_n)$, $b = b(x_1, x_2, \ldots, x_n)$, such that

$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F)) \leq (1-p)/2$ holds.

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$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F)) \leq (1-p)/2$ holds.

Assume w.l.o.g. that the sample is sorted $x_1 \geq x_2 \geq \ldots \geq x_n$.

Determine $i > j$, $i, j \in \{1, 2, \ldots, n\}$, and the smallest $p' > p$, such that

$$P\left(x_i < q_\alpha(F) < x_j\right) = p' \qquad (*) \qquad \text{and}$$

$$P\left(x_i \geq q_\alpha(F)\right) \leq (1-p)/2 \text{ and } P\left(x_j \leq q_\alpha(F)\right) \leq (1-p)/2 (**).$$

## An approximative solution without bootstrapping (contd.)

Let $Y_\alpha := \#\{x_k : x_k > q_\alpha(F)\}$

**An approximative solution without bootstrapping (contd.)**

Let $Y_\alpha := \#\{x_k\colon x_k > q_\alpha(F)\}$

We get $P(x_j \leq q_\alpha(F)) \approx P(x_j < q_\alpha(F)) = P(Y_\alpha \leq j-1)$
$P(x_i \geq q_\alpha(F)) \approx P(x_i > q_\alpha(F)) = 1 - P(Y_\alpha \leq i-1)$

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$Y_\alpha \sim Bin(n, 1 - \alpha)$ since $Prob(x_k \geq q_\alpha(F)) \approx 1 - \alpha$ for a sample point $x_k$.

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Compute $P(x_j \leq q_\alpha(F))$ and $P(x_i \geq q_\alpha(F))$ for different $i$ and $j$ until indices $i, j \in \{1, 2, \ldots, n\}$, $i > j$, which fulfill $(**)$ are found.

**An approximative solution without bootstrapping (contd.)**

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Set $b := x_j$ and $a := x_i$.