Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

Possibilities to generate a sample of losses $x_1,\ldots,x_n$

(i) Historical simulation
Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

Assumption: the historically realized losses are i.i.d.
The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses: $l_1 \geq l_2 \geq \ldots \geq l_n$.

Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

Assumption: the historically realized losses are i.i.d.
The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses: $l_1 \geq l_2 \geq \ldots \geq l_n$.

Empirical VaR: $\widehat{VaR} = q_\alpha(\hat{F}_n^L) = l_{[n(1-\alpha)]+1}$

Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

Assumption: the historically realized losses are i.i.d.
The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses: $l_1 \geq l_2 \geq \ldots \geq l_n$.

Empirical VaR: $\widehat{VaR} = q_\alpha(\hat{F}_n^L) = l_{[n(1-\alpha)]+1}$

Empirical CVaR: $\widehat{CVaR} = \frac{\sum_{i=1}^{[n(1-\alpha)]+1} l_i}{[n(1-\alpha)]+1}$.

Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation
Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$;
the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

Assumption: the historically realized losses are i.i.d.
The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses: $l_1 \geq l_2 \geq \ldots \geq l_n$.

Empirical VaR: $\widehat{VaR} = q_\alpha(\hat{F}_n^L) = l_{[n(1-\alpha)]+1}$

Empirical CVaR: $\widehat{CVaR} = \frac{\sum_{i=1}^{[n(1-\alpha)]+1} l_i}{[n(1-\alpha)]+1}$.

The aggregated loss over a given time interval
For example, for 10 time units, compute $\lfloor n/10 \rfloor$ aggregated loss realizations $l_k^{(10)}$ over the time intervals
$[m-n+10(k-1)+1, m-n+10(k-1)+10]$, $k = 1, \ldots, \lfloor n/10 \rfloor$:
$l_k^{(10)} = l_{[m]}\left(\sum_{j=1}^{10} x_{m-n+10(k-1)+j}\right)$.
Then compute the empirical estimators of the risk measures.

Historical simulation (contd.)

Advantages:

- simple implementation
- considers intrinsically the dependencies between the elements of the vector of the risk factors changes $X_{m-k} = (X_{m-k,1}, \ldots, X_{m-k,d})$.

Historical simulation (contd.)

Advantages:

- simple implementation
- considers intrinsically the dependencies between the elements of the vector of the risk factors changes $X_{m-k} = (X_{m-k,1}, \dots, X_{m-k,d})$.

Disadvantages:

- lots of historical data needed to get good estimators
- the estimated loss cannot be larger than the maximal loss experienced in the past

(ii) The variance-covariance method

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_m(X_{m+1}) = -V\sum_{i=1}^d w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

Let $x_{m-n+1}, \ldots, x_m$ be the historically observed risk factor changes

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

Let $x_{m-n+1}, \ldots, x_m$ be the historically observed risk factor changes

Assumption 2: $x_{m-n+1}, \ldots, x_m$ are i.i.d.

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_m(X_{m+1}) = -V\sum_{i=1}^d w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

Let $x_{m-n+1}, \ldots, x_m$ be the historically observed risk factor changes

Assumption 2: $x_{m-n+1}, \ldots, x_m$ are i.i.d.

Estimator for μ_i: $\hat{\mu}_i = \frac{1}{n}\sum_{k=1}^n x_{m-k+1,i}$, $i = 1, 2, \ldots, d$

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L_{m+1}^{\Delta} = l_m^{\Delta}(X_{m+1}) = -V \sum_{i=1}^{d} w_i X_{m+1,i} = -V w^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-V w^T X_{m+1} \sim N(-V w^T \mu, V^2 w^T \Sigma w)$

Let $x_{m-n+1}, \ldots, x_m$ be the historically observed risk factor changes

Assumption 2: $x_{m-n+1}, \ldots, x_m$ are i.i.d.

Estimator for μ_i: $\hat{\mu}_i = \frac{1}{n} \sum_{k=1}^{n} x_{m-k+1,i}$, $i = 1, 2, \ldots, d$

Estimator for $\Sigma = \left(\sigma_{ij} \right)$: $\hat{\Sigma} = \left(\hat{\sigma}_{ij} \right)$ with

$\hat{\sigma}_{ij} = \frac{1}{n-1} \sum_{k=1}^{n} (x_{m-k+1,i} - \mu_i)(x_{m-k+1,j} - \mu_j) \qquad i, j = 1, 2, \ldots, d$

(ii) The variance-covariance method

Idea: use the linearised loss function under the assumption that the vector of the risk factor changes is normally distributed.

$L^{\Delta}_{m+1} = l^{\Delta}_m(X_{m+1}) = -V\sum_{i=1}^d w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

Assumption 1: $X_{m+1} \sim N_d(\mu, \Sigma)$,

and thus $-Vw^T X_{m+1} \sim N(-Vw^T\mu, V^2 w^T \Sigma w)$

Let $x_{m-n+1}, \ldots, x_m$ be the historically observed risk factor changes

Assumption 2: $x_{m-n+1}, \ldots, x_m$ are i.i.d.

Estimator for μ_i: $\hat{\mu}_i = \frac{1}{n}\sum_{k=1}^n x_{m-k+1,i}$, $i = 1, 2, \ldots, d$

Estimator for $\Sigma = \Big(\sigma_{ij}\Big)$: $\hat{\Sigma} = \Big(\hat{\sigma}_{ij}\Big)$ with

$\hat{\sigma}_{ij} = \frac{1}{n-1}\sum_{k=1}^n (x_{m-k+1,i} - \mu_i)(x_{m-k+1,j} - \mu_j) \qquad i, j = 1, 2, \ldots, d$

Estimator for VaR: $\widehat{VaR}(L_{m+1}) = -Vw^T\hat{\mu} + V\sqrt{w^T\hat{\Sigma}w}\,\phi^{-1}(\alpha)$

The variance-covariance method (contd.)

Advantages:

- analytical solution
- simple implementation
- no simulations needed

The variance-covariance method (contd.)

Advantages:

- analytical solution
- simple implementation
- no simulations needed

Disadvantages:

- Linearisation is not always appropriate, only for a short time horizon reasonable
- The normal distribution assumption could lead to underestimation of risks and should be argued upon (e.g. in terms of historical data)

(iii) Monte-Carlo approach

(1) historical observations of risk factor changes $X_{m-n+1}, \ldots, X_m$.

(2) assumption on a parametric model for the cumulative distribution function of X_k, $m-n+1 \leq k \leq m$;
e.g. a common distribution function F and independence

(3) estimation of the parameters of F.

(4) generation of N samples $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ from F $(N \gg 1)$ and computation of the losses $l_k = l_{[m]}(\tilde{x}_k)$, $1 \leq k \leq N$

(5) computation of the empirical distribution of the loss function L_{m+1}:

$$\hat{F}_N^{L_{m+1}}(x) = \frac{1}{N} \sum_{k=1}^{N} I_{[l_k,\infty)}(x).$$

(5) computation of estimates for the VaR and CVAR of the loss function: $\widehat{VaR}(L_{m+1}) = (\hat{F}_N^{L_{m+1}}\Big) = l_{[N(1-\alpha)]+1}$,

$\widehat{CVaR}(L_{m+1}) = \frac{\sum_{k=1}^{[N(1-\alpha)]+1} l_k}{[N(1-\alpha)]+1}$,

where the losses are sorted as $l_1 \geq l_2 \geq \ldots \geq l_N$.

Monte-Carlo approach (contd.)

Advantages:

- very flexible; can use any distribution F from which simulation is possible
- time dependencies of the risk factor changes can be considered by using time series

Monte-Carlo approach (contd.)

Advantages:

- very flexible; can use any distribution F from which simulation is possible
- time dependencies of the risk factor changes can be considered by using time series

Disadvantages:

- computationally expensive; a large number of simulations needed to obtain good estimates

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

θ can be estimated by means of historical data (e.g. maximum likelihood approaches)

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

θ can be estimated by means of historical data (e.g. maximum likelihood approaches)

Let the price at time t_k be $S := S_{t_k}$

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

θ can be estimated by means of historical data (e.g. maximum likelihood approaches)

Let the price at time t_k be $S := S_{t_k}$

The VaR of the portfolio over $[t_k, t_{k+1}]$ is given as

$$VaR_\alpha(L_{t_k+1}) = S\left(1 - \exp\{F_\theta^{\leftarrow}(1-\alpha)\}\right).$$

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

θ can be estimated by means of historical data (e.g. maximum likelihood approaches)

Let the price at time t_k be $S := S_{t_k}$

The VaR of the portfolio over $[t_k, t_{k+1}]$ is given as

$$VaR_\alpha(L_{t_k+1}) = S\left(1 - \exp\{F_\theta^{\leftarrow}(1-\alpha)\}\right).$$

Depending on F_θ it can be complicated or impossible to compute CVaR analytically.

Monte-Carlo approach

Example: The portfolio consists of one unit of asset S with price S_t at time t. The risk factor changes $X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k})$ are i.i.d. with distribution function F_θ for some unknown parameter θ.

θ can be estimated by means of historical data (e.g. maximum likelihood approaches)

Let the price at time t_k be $S := S_{t_k}$

The VaR of the portfolio over $[t_k, t_{k+1}]$ is given as

$$VaR_\alpha(L_{t_k+1}) = S\left(1 - \exp\{F_\theta^{\leftarrow}(1-\alpha)\}\right).$$

Depending on F_θ it can be complicated or impossible to compute CVaR analytically.

Alternative: Monte-Carlo simulation.

Example (contd.)

A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

Example (contd.)

A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

$$
\begin{aligned}
X_{k+1} &= \sigma_{k+1} Z_{k+1} && (1)\\
\sigma_{k+1}^2 &= a_0 + a_1 X_k^2 + b_1 \sigma_k^2 && (2)
\end{aligned}
$$

where Z_k, $k \in \mathbb{N}$, are i.i.d. and standard normally distributed, and a_0,a_1 and b_1 are parameters, which should be estimated.

Example (contd.)

A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

$$\begin{aligned} X_{k+1} &= \sigma_{k+1} Z_{k+1} && (1) \\ \sigma_{k+1}^2 &= a_0 + a_1 X_k^2 + b_1 \sigma_k^2 && (2) \end{aligned}$$

where Z_k, $k \in \mathbb{N}$, are i.i.d. and standard normally distributed, and a_0,a_1 and b_1 are parameters, which should be estimated.

It is simple to simulate from this model.

Example (contd.)

A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

$$\begin{aligned} X_{k+1} &= \sigma_{k+1} Z_{k+1} && (1) \\ \sigma_{k+1}^2 &= a_0 + a_1 X_k^2 + b_1 \sigma_k^2 && (2) \end{aligned}$$

where Z_k, $k \in \mathbb{N}$, are i.i.d. and standard normally distributed, and a_0,a_1 and b_1 are parameters, which should be estimated.

It is simple to simulate from this model.

Howeve, analytical computation of VaR and CVaR over a certain time interval consisting of many periods is cumbersome! Check it out!

Chapter 3: Extreme value theory

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
- $\bar{F} := 1 - F$ is called the *right tail* of the univariate distribution function F.

Chapter 3: Extreme value theory

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
- $\bar{F} := 1 - F$ is called the *right tail* of the univariate distribution function F.

Terminology: We say a r.v. X has **fat tails** or is **heavy tailed** (h.t.) iff $\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty$, $\forall \lambda > 0$.

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
- $\bar{F} := 1 - F$ is called the *right tail* of the univariate distribution function F.

Terminology: We say a r.v. X has **fat tails** or is **heavy tailed** (h.t.) iff $\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty$, $\forall \lambda > 0$.

Also a r.v. X for which $\exists k \in \mathbb{N}$ with $E(X^k) = \infty$ will be often called **heavy tailed**.

Chapter 3: Extreme value theory

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
- $\bar{F} := 1 - F$ is called the *right tail* of the univariate distribution function F.

Terminology: We say a r.v. X has **fat tails** or is **heavy tailed** (h.t.) iff $\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty$, $\forall \lambda > 0$.

Also a r.v. X for which $\exists k \in \mathbb{N}$ with $E(X^k) = \infty$ will be often called **heavy tailed**.

These two "definitions" are not equivalent!

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t \to +\infty} \frac{h(tx)}{h(t)} = x^{\rho}\,,\ \forall x > 0 \tag{3}$$

Notation: $h \in RV_{\rho}$.

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t\to+\infty} \frac{h(tx)}{h(t)} = x^{\rho}\ ,\ \forall x > 0 \tag{3}$$

Notation: $h \in RV_{\rho}$.

If $\rho = 0$, *we say* h **has a slow variation** *or* **is slowly varying** *towards* ∞.

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t \to +\infty} \frac{h(tx)}{h(t)} = x^{\rho}\,,\ \forall x > 0 \tag{3}$$

Notation: $h \in RV_{\rho}$.

If $\rho = 0$, *we say* h **has a slow variation** *or* **is slowly varying** *towards* ∞.
If $h \in RV_{\rho}$, *then* $h(x)/x^{\rho} \in RV_0$, *or equivalently,*

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t\to+\infty} \frac{h(tx)}{h(t)} = x^{\rho}\,,\ \forall x > 0 \tag{3}$$

Notation: $h \in RV_{\rho}$.

If $\rho = 0$, *we say* h **has a slow variation** *or* **is slowly varying** *towards* ∞.
If $h \in RV_{\rho}$, *then* $h(x)/x^{\rho} \in RV_0$, *or equivalently,*
if $h \in RV_{\rho}$, *then* $\exists L \in RV_0$ *such that* $h(x) = L(x)x^{\rho}$ $(L(x) = h(x)/x^{\rho})$.

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t\to+\infty} \frac{h(tx)}{h(t)} = x^{\rho}\,,\ \forall x > 0 \tag{3}$$

Notation: $h \in RV_{\rho}$.

If $\rho = 0$, *we say* h **has a slow variation** *or* **is slowly varying** *towards* ∞.
If $h \in RV_{\rho}$, *then* $h(x)/x^{\rho} \in RV_0$, *or equivalently,*
if $h \in RV_{\rho}$, *then* $\exists L \in RV_0$ *such that* $h(x) = L(x)x^{\rho}$ $(L(x) = h(x)/x^{\rho})$.
If $\rho < 0$, *then the convergence in (3) is uniform in every interval* $(b,+\infty)$ *for* $b > 0$.

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has* **a regular variation with index** $\rho \in \mathbb{R}$ **towards** $+\infty$ *iff*

$$\lim_{t\to+\infty} \frac{h(tx)}{h(t)} = x^{\rho}\,, \ \forall x > 0 \tag{3}$$

Notation: $h \in RV_\rho$.

If $\rho = 0$, *we say* h **has a slow variation** *or* **is slowly varying** *towards* ∞.
If $h \in RV_\rho$, *then* $h(x)/x^\rho \in RV_0$, *or equivalently,*
if $h \in RV_\rho$, *then* $\exists L \in RV_0$ *such that* $h(x) = L(x)x^\rho$ $(L(x) = h(x)/x^\rho)$.
If $\rho < 0$, *then the convergence in (3) is uniform in every interval* $(b,+\infty)$ *for* $b > 0$.

Example

Show that $L \in RV_0$ *holds for the functions* L *as below:*

(a) $\lim_{x\to+\infty} L(x) = c \in (0,+\infty)$

(b) $L(x) := \ln(1+x)$

(c) $L(x) := \ln(1+\ln(1+x))$

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\liminf_{x\to\infty} L(x) = 0 \text{ and } \limsup_{x\to\infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x \to \infty} L(x) = 0 \text{ and } \lim \sup_{x \to \infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

Definition: A **r.v.** $X > 0$ **with distribution function** F **has a regular variation on** $+\infty$, iff $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$.

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x \to \infty} L(x) = 0 \text{ and } \lim \sup_{x \to \infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

Definition: A **r.v.** $X > 0$ **with distribution function** F **has a regular variation on** $+\infty$, iff $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$.

Example:

1. Pareto distribution: $G_\alpha(x) := 1 - x^{-\alpha}$, for $x > 1$ and $\alpha > 0$. Then $\bar{G}_\alpha(tx)/\bar{G}_\alpha(x) = x^{-\alpha}$ holds for $t > 0$, i.e. $\bar{G}_\alpha \in RV_{-\alpha}$.

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x\to\infty} L(x) = 0 \text{ and } \lim \sup_{x\to\infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

Definition: A **r.v.** $X > 0$ **with distribution function** F **has a regular variation on** $+\infty$, iff $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$.

Example:

1. Pareto distribution: $G_\alpha(x) := 1 - x^{-\alpha}$, for $x > 1$ and $\alpha > 0$. Then $\bar{G}_\alpha(tx)/\bar{G}_\alpha(x) = x^{-\alpha}$ holds for $t > 0$, i.e. $\bar{G}_\alpha \in RV_{-\alpha}$.
2. Fréchet distribution: $\Phi_\alpha(x) := \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, for some parameter (fixed) $\alpha > 0$. Then $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$ holds, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x\to\infty} L(x) = 0 \text{ and } \lim \sup_{x\to\infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1+x))^2 \cos((\ln(1+x))^{1/2})\}$.

Definition: A **r.v.** $X > 0$ **with distribution function** F **has a regular variation on** $+\infty$, iff $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$.

Example:

1. Pareto distribution: $G_\alpha(x) := 1 - x^{-\alpha}$, for $x > 1$ and $\alpha > 0$. Then $\bar{G}_\alpha(tx)/\bar{G}_\alpha(x) = x^{-\alpha}$ holds for $t > 0$, i.e. $\bar{G}_\alpha \in RV_{-\alpha}$.
2. Fréchet distribution: $\Phi_\alpha(x) := \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, for some parameter (fixed) $\alpha > 0$. Then $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$ holds, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

Proposition (no proof)
Let $X > 0$ be a r.v. with distribution function F, such that $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Then $E(X^\beta) < \infty$ for $\beta < \alpha$ and $E(X^\beta) = \infty$ for $\beta > \alpha$ hold.

Example: Check whether $f \in RV_0$ holds for $f(x) = 3 + \sin x$, $f(x) = \ln(e + x) + \sin x$?

Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x \to \infty} L(x) = 0 \text{ and } \lim \sup_{x \to \infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

Definition: A **r.v.** $X > 0$ **with distribution function** F **has a regular variation on** $+\infty$, iff $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$.

Example:

1. Pareto distribution: $G_\alpha(x) := 1 - x^{-\alpha}$, for $x > 1$ and $\alpha > 0$. Then $\bar{G}_\alpha(tx)/\bar{G}_\alpha(x) = x^{-\alpha}$ holds for $t > 0$, i.e. $\bar{G}_\alpha \in RV_{-\alpha}$.
2. Fréchet distribution: $\Phi_\alpha(x) := \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, for some parameter (fixed) $\alpha > 0$. Then $\lim_{x \to \infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$ holds, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

Proposition (no proof)
Let $X > 0$ be a r.v. with distribution function F, such that $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Then $E(X^\beta) < \infty$ for $\beta < \alpha$ and $E(X^\beta) = \infty$ for $\beta > \alpha$ hold.
The converse is not true!

Application of regular variation

Example 1: Let X_1 and X_2 be two continuous nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

Application of regular variation

Example 1: Let X_1 and X_2 be two continuous nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

Assumption: The prices of A_1 and A_2 are identical and their logreturns are i.i.d..

Application of regular variation

Example 1: Let X_1 and X_2 be two continuous nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

Assumption: The prices of A_1 and A_2 are identical and their logreturns are i.i.d..

Consider a portfolio P_1 containing 2 units of asset A_1 and a portfolio P_2 containing one unit of A_1 and one unit of A_2. Let L_i represent the loss of portfolio P_i, $i = 1, 2$.

Application of regular variation

Example 1: Let X_1 and X_2 be two continuous nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

Assumption: The prices of A_1 and A_2 are identical and their logreturns are i.i.d..

Consider a portfolio P_1 containing 2 units of asset A_1 and a portfolio P_2 containing one unit of A_1 and one unit of A_2. Let L_i represent the loss of portfolio P_i, $i = 1, 2$.

Compare the probabilities of high losses in the two portfolios by computing the limit

$$\lim_{l \to \infty} \frac{Prob(L_2 > l)}{Prob(L_1 > l)} .$$

In which cases are the extreme losses of the diversified portfolio smaller then those of the non-diversified portfolio?

Application of regular variation (contd.)

Example 2: Let $X, Y \geq 0$ be two r.v. which represent the losses of two business lines of an insurance company (e.g. fire and car accidents).

Application of regular variation (contd.)

Example 2: Let $X, Y \geq 0$ be two r.v. which represent the losses of two business lines of an insurance company (e.g. fire and car accidents).

Assumptions

- $\bar{F} \in RV_{-\alpha}$, for some $\alpha > 0$, where F is the distribution function of X.
- $E(Y^k) < \infty$, $\forall k > 0$.

Application of regular variation (contd.)

Example 2: Let $X, Y \geq 0$ be two r.v. which represent the losses of two business lines of an insurance company (e.g. fire and car accidents).

Assumptions

- $\bar{F} \in RV_{-\alpha}$, for some $\alpha > 0$, where F is the distribution function of X.
- $E(Y^k) < \infty$, $\forall k > 0$.

Compute $\lim_{x \to \infty} P(X > x | X + Y > x)$.