Classical extreme value theory

Let (X_k), $k \in \mathbb{N}$, be non-degenerate i.i.d. r.v. with distribution function F. For $n \geq 1$ define $S_n := \sum_{i=1}^{n} X_i$ and $M_n := \max\{X_i \colon 1 \leq i \leq n\}$

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Definition: A r.v. X is called **stable, (α-stable, Lévy-stable)**, iff for all $c_1, c_2 \in \mathbb{R}_+$ and the i.i.d. copies X_1 and X_2 of X, there exist constantes $a(c_1, c_2) \in \mathbb{R}$ and $b(c_1, c_2) \in \mathbb{R}$, such that $c_1 X_1 + c_2 X_2$ und $a(c_1, c_2)X + b(c_1, c_2)$ are identically distributed.
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Theorem

The family of stable distributions coincides whith the limit distributions of appropriately normalized and centralized sums of i.i.d. r.v..

Proof e.g. in Rényi, 1962.

Stable distributions (contd.)

Theorem: The characteristic function of a stable distribution X is given as:

$$\varphi_X(t) = E[\exp\{iXt\}] = \exp\{i\gamma t - c|t|^\alpha(1 + i\beta \mathrm{signum}(t) z(t,\alpha))\}\,, \quad (4)$$

where $\gamma \in \mathbb{R}$, $c > 0$, $\alpha \in (0,2]$, $\beta \in [-1,1]$ and

$$z(t,\alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{wenn } \alpha \neq 1 \\ -\frac{2}{\pi}\ln|t| & \text{wenn } \alpha = 1 \end{cases}$$

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Definition: The parameter α in (4) is called **the form parameter or the characteristical exponent**, the corresponding distribution is called α-stable and its distribution function is denoted by G_α.

Definition: Let X be a r.v. with distribution function F. Assume that there exists two sequences of reals $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n^{-1}(S_n - b_n) = G_\alpha$, for some α-stable distribution G_α. Then we say that F **belongs to the domain of attraction of** G_α.
Notation: $F \in DA(G_\alpha)$.

Stable distributions (contd.)

Remark 1:

$X \sim G_2 \iff \varphi_X(t) = \exp\{i\gamma t - \frac{1}{2}t^2(2c)\} \iff X \sim N(\gamma, 2c)$

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Remark2: Show that $F \in DA(G_2) \iff F \in DA(\phi)$, where ϕ is the standard normal distribution $N(0,1)$.
Hint: The Convergence to Types Theorem could be used.

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Hint: The Convergence to Types Theorem could be used.

Definition: The r.v. Z and $\tilde{Z}$ **are of the same type** if there exist the constants $\sigma > 0$ and $\mu \in \mathbb{R}$, such that $\tilde{Z} \stackrel{\mathrm{d}}{=} (Z - \mu)/\sigma$, i.e. $\tilde{F}(x) = F(\mu + \sigma x)$, $\forall x \in \mathbb{R}$, where F and $\tilde{F}$ are the distribution functions of Z and $\tilde{Z}$, respectively.

The Convergence to Types Theorem

Let Z, $\tilde{Z}$, Y_n, $n \geq 1$, be not almost surely constant r.v.
Let $a_n, \tilde{a}_n, b_n, \tilde{b}_n \in \mathbb{R}$, $n \in \mathbb{N}$, be sequences of reals with $a_n, \tilde{a}_n > 0$.

(i) If

$$\lim_{n\to\infty} a_n^{-1}(Y_n - b_n) = Z \text{ and } \lim_{n\to\infty} \tilde{a}_n^{-1}(Y_n - \tilde{b}_n) = \tilde{Z} \qquad (5)$$

then there exist $A > 0$ und $B \in \mathbb{R}$, such that

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(ii) Assume that (6) holds. Then each of the two relations in (5) implies the other and also (7) holds.

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Proof: See Resnick 1987, Prop. 0.2, Seite 7.

Characterization of the domain of attraction

(i) Let ϕ be the standard normal distribution function. The equivalence

$$F \in DA(\phi) \Longleftrightarrow \lim_{x\to\infty} \frac{x^2 \int_{[-x,x]^C} dF(y)}{\int_{[-x,x]} y^2 dF(y)} = 0$$

holds, where $[-x,x]^C$ is the complement of $[-x,x]$ in $\mathbb{R}$.

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$$F \in DA(G_\alpha) \Longleftrightarrow F(-x) = \frac{c_1 + o(1)}{x^\alpha} L(x), \bar{F}(x) = \frac{c_2 + o(1)}{x^\alpha} L(x)$$

holds, where L is a slowly varying function around infinity and $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

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Remark: Let $F \in DA(G_\alpha)$ for $\alpha \in (0,2)$. Then $E(|X|^\delta) < \infty$ for $\delta < \alpha$ and $E(|X|^\delta) = \infty$ for $\delta > \alpha$ hold.

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Proof: See Resnick 1987 (or a demanding homework!)

Limit distributions of normalized and centered maxima

Let (X_k), $k \in \mathbb{N}$, be non-degenerate i.i.d. r.v. with distribution function F.

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Theorem: (Poisson Approximation)
Let $\tau \in [0, \infty]$ and a sequence of reals $u_n \in \mathbb{R}$. Then the following holds

$$\lim_{n\to\infty} n\bar{F}(u_n) = \tau \iff \lim_{n\to\infty} P(M_n \leq u_n) = exp\{-\tau\}.$$

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Remark: The convergence to types theorem implies that H and $\tilde{H}$ are of the same type, if
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Max-stable distributions and the Fischer-Tippet theorem

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Definition: A non-degenarate r.v. X is called *max-stable* iff for any $n \geq 2$ $\max\{X_1, X_2, \ldots, X_n\} \stackrel{\mathrm{d}}{=} a_n X + b_n$ for indepedent copies X_1,X_2, ...,X_n of X and appropriate constants $a_n > 0$ and $b_n \in \mathbb{R}$.

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Theorem: (Fischer-Tippet Theorem, Proof in Resnick 1987, page 9-11)
Let (X_k) be a sequence of i.i.d. r.v.. If the constants $a_n, b_n \in \mathbb{R}$, $a_n > 0$, and a non-degenerate disribution H exist, such that $\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$, then H is of the same type as one of the following three distributions:

$$\begin{array}{lll} \text{Fréchet} & \Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp\{-x^{-\alpha}\} & x > 0 \end{cases} & \alpha > 0 \\ \text{Weibull} & \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x \leq 0 \\ 1 & x > 0 \end{cases} & \alpha > 0 \\ \text{Gumbel} & \Lambda(x) = \exp\{-e^{-x}\} & x \in \mathbb{R} \end{array}$$

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The distributions Φ_α, Ψ_α and Λ are called *standard extreme value distributions (standard evd)*. The distributions which are of the same type as the standard evd are called *extreme value distributions* (evd).

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Theorem: (Characterisation of MDA, proof is left as an exercise) $F \in MDA(H)$ with normalizing and centering constants $a_n > 0$ snd $b_n \in \mathbb{R}$ holds, iff

$$\lim_{n\to\infty} n\bar{F}(a_n x + b_n) = -\ln H(x), \forall x \in \mathbb{R},$$

where $-\ln H(x)$ is replaced by ∞ if $H(x) = 0$.

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There exist distributions which do not belong to the MDA of any evd!
Example: (The Poisson distribution)
Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, $\lambda > 0$. Show that there exist no evd Z such that $X \in MDA(Z)$.

The generalized evd

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Definition: (The generalized extreme value distribution (gevd))
Let the distribution function H_γ be given as follows:

$$H_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{wenn } \gamma \neq 0 \\ \exp\{-\exp\{-x\}\} & \text{wenn } \gamma = 0 \end{cases}$$

where $1 + \gamma x > 0$, i.e. the definition area of H_γ is given as

$$\begin{array}{ll} x > -\gamma^{-1} & \text{wenn } \gamma > 0 \\ x < -\gamma^{-1} & \text{wenn } \gamma < 0 \\ x \in \mathbb{R} & \text{wenn } \gamma = 0 \end{array}$$

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The generalized evd

Definition: (The generalized extreme value distribution (gevd))
Let the distribution function H_γ be given as follows:

$$H_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{wenn } \gamma \neq 0 \\ \exp\{-\exp\{-x\}\} & \text{wenn } \gamma = 0 \end{cases}$$

where $1 + \gamma x > 0$, i.e. the definition area of H_γ is given as

$$\begin{array}{ll} x > -\gamma^{-1} & \text{wenn } \gamma > 0 \\ x < -\gamma^{-1} & \text{wenn } \gamma < 0 \\ x \in \mathbb{R} & \text{wenn } \gamma = 0 \end{array}$$

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Theorem: (Characterisation of $MDA(H_\gamma)$)

- $F \in MDA(H_\gamma)$ with $\gamma > 0 \iff F \in MDA(\Phi_\alpha)$ with $\alpha = 1/\gamma > 0$.
- $F \in MDA(H_0) \iff F \in MDA(\Lambda)$.
- $F \in MDA(H_\gamma)$ with $\gamma < 0 \iff F \in MDA(\Psi_\alpha)$ with $\alpha = -1/\gamma > 0$.

MDAs: Examples and Characterisations

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Examples: The following distributions belong to $MDA(\Phi_\alpha)$:

- Pareto: $F(x) = 1 - x^{-\alpha}$, $x > 1$, $\alpha > 0$.
- Cauchy: $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.
- Student: $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.
- Loggamma: $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} x^{-\alpha-1}$, $x > 1$, $\alpha, \beta > 0$.