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If $F \in MDA(\Psi_\alpha)$, then $\lim_{n\to\infty} a_n^{-1}(M_n - x_F) = \Psi_\alpha$ with $a_n = x_F - F^{\leftarrow}(1 - n^{-1})$.

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Example: Let $X \sim U(0,1)$. it holds $X \in MDA(\Psi_1)$ with $a_n = 1/n$, $n \in \mathbb{N}$.

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Theorem: $(MDA(\Lambda))$
Let F be a distribution function with right endpoint $x_F \leq \infty$.
$F \in MDA(\Lambda)$ holds iff there exists a $z < x_F$ such that F can be represented as

$$\bar{F}(x) = c(x) exp\left\{ - \int_z^x \frac{g(t)}{a(t)} dt \right\}, \forall x, z < x \leq x_F,$$

where the functions $c(x)$ and $g(x)$ fulfill $\lim_{x \uparrow x_F} c(x) = c > 0$ and $\lim_{t \uparrow x_F} g(t) = 1$, and $a(t)$ is a positive absolutely continuous function with $\lim_{t \uparrow x_F} a'(t) = 0$.

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See the book by Embrechts et al. for the proofs of the above theorem and of the following theorem concerning the characterisation of $MDA(\Lambda)$.

Characterisations of MDAs (contd.)

Theorem: ($MDA(\Lambda)$, alternative characterisation)
A distribution function F belongs to $MDA(\Lambda)$ iff there exists a positive function $\tilde{a}$ such that

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x + u\tilde{a}(x))}{\bar{F}(x)} = e^{-u}, \forall u \in \mathbb{R}$$

A possible choice for $\tilde{a}$ is $\tilde{a}(x) = a(x)$ with $a(x) := \int_x^{x_F} \frac{\bar{F}(t)}{\bar{F}(x)} dt$.

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Examples: The following distributions belong to $MDA(\Lambda)$:

- Normal: $F(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$, $x \in \mathbb{R}$.
- Exponential: $f(x) = \lambda^{-1} \exp\{-\lambda x\}$, $x > 0$, $\lambda > 0$.
- Lognormal: $f(x) = (2\pi x^2)^{-1/2} \exp\{-(\ln x)^2/2\}$, $x > 0$.
- Gamma: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-\beta x\}$, $x > 0$, $\alpha, \beta > 0$.

Graphical methods for the investigation of the right tail of the distribution

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- Quantile-quantile plots

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  Question: How to check whether this assumption holds?

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  Let $x_n \leq x_{n-1} \leq \ldots \leq x_1$ be a sorted sample of X_1, X_2,..., X_n.
  qq-plot: $\{(x_k, F^{\leftarrow}(\frac{n-k+1}{n+1})) \colon k = 1, 2, \ldots, n\}$.

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  Rule of thumb: the larger the quantile the heavier the tails of the distribution!

The Hill estimator

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Goal: Estimate α!

Theorem: (Theorem of Karamata)
Let L be a slowly varying locally bounded function on $[x_0, +\infty)$ for some $x_0 \in \mathbb{R}$. Then the following holds:

(a) For $\kappa > -1$: $\int_{x_o}^{x} t^{\kappa} L(t) dt \sim K(x_0) + \frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$, where $K(x_0)$ is a constant depending on x_0.

(b) For $\kappa < -1$: $\int_{x}^{+\infty} t^{\kappa} L(t) dt \sim -\frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$.

Proof in Bingham et al. 1987.

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The theorem of Karamata implies: $E\left(\ln(X) - \ln(u) | \ln(X) > \ln(u)\right) =$

$$\lim_{u \to \infty} \frac{1}{\bar{F}(u)} \int_u^{\infty} (\ln x - \ln u) dF(x) = \alpha^{-1}. \tag{8}$$

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For the empirical distribution $F_n(x) = \frac{1}{n} \sum_{k=1}^n I_{[x_k, \infty)}(x)$ and a large threshold x_k depending on the sample $x_n \leq x_{n-1} \leq \ldots \leq x_1$ we get:

$$E\left(\ln(X) - \ln(x_k) | \ln(X) > \ln(x_k)\right) \approx$$

$$\frac{1}{\bar{F}_n(x_k)} \int_{X_k}^\infty (\ln x - \ln x_k) dF_n(x) = \frac{1}{k-1} \sum_{j=1}^{k-1} (\ln x_j - \ln x_k).$$

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If $k = k(n) \to \infty$ and $k/n \to 0$, then $x_k \to \infty$ for $n \to \infty$, and (8) implies:

$$\lim_{n\to\infty} \frac{1}{k-1} \sum_{j=1}^{k-1} (\ln x_j - \ln x_k) \stackrel{d}{=} \alpha^{-1}$$

Hill estimators for the tail distribution and the quantile

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Given an estimator $\hat{\alpha}_{k,n}^{(H)}$ of α we get tail and quantile estimators as follows:

$$\hat{\bar{F}}(x) = \frac{k}{n} \left(\frac{x}{x_k} \right)^{-\hat{\alpha}_{k,n}^{(H)}} \text{ and } \hat{q}_p = \hat{F}^{\leftarrow}(p) = \left(\frac{n}{k}(1-p) \right)^{-1/\hat{\alpha}_{k,n}^{(H)}} x_k.$$

The POT method (Peaks over Threshold)

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Definition: (The generalized Pareto distribution (GPD))
The **standard GPD** denoted by G_γ:

$$G_\gamma(x) = \begin{cases} 1 - (1+\gamma x)^{-1/\gamma} & \text{für } \gamma \neq 0 \\ 1 - \exp\{-x\} & \text{für } \gamma = 0 \end{cases}$$

where $x \in D(\gamma)$

$$D(\gamma) = \begin{cases} 0 \leq x < \infty & \text{für } \gamma \geq 0 \\ 0 \leq x \leq -1/\gamma & \text{für } \gamma < 0 \end{cases}$$

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Notice that $G_0 = \lim_{\gamma \to 0} G_\gamma$.

Let $\nu \in \mathbb{R}$ and $\beta > 0$. The **GPD** with parameters γ, ν, β is given by the following distribution function

$$G_{\gamma,\nu,\beta} = 1 - (1 + \gamma \frac{x-\nu}{\beta})^{-1/\gamma}$$

where $x \in D(\gamma, \nu, \beta)$ and

$$D(\gamma,\nu,\beta) = \begin{cases} \nu \leq x < \infty & \text{für } \gamma \geq 0 \\ \nu \leq x \leq \nu - \beta/\gamma & \text{für } \gamma < 0 \end{cases}$$