Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho\colon M \to \mathbb{R}$ a risk measure in M.

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho \colon M \to \mathbb{R}$ a risk measure in M.

Definition: A risk measure ρ in M is called *coherent* iff it has the following properties

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho\colon M \to \mathbb{R}$ a risk measure in M.

Definition: A risk measure ρ in M is called *coherent* iff it has the following properties

(C1) Invariance with respect to translation:

$\rho(X + r) = \rho(X) + r$, $\forall r \in \mathbb{R}$ and $\forall X \in M$.

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho \colon M \to \mathbb{R}$ a risk measure in M.

Definition: A risk measure ρ in M is called *coherent* iff it has the following properties

(C1) Invariance with respect to translation:

$\rho(X + r) = \rho(X) + r$, $\forall r \in \mathbb{R}$ and $\forall X \in M$.

(C2) Subadditivity:

$\forall X_1, X_2 \in M$ it holds $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho\colon M \to \mathbb{R}$ a risk measure in M.

Definition: A risk measure ρ in M is called *coherent* iff it has the following properties

(C1) Invariance with respect to translation:

$\rho(X + r) = \rho(X) + r$, $\forall r \in \mathbb{R}$ and $\forall X \in M$.

(C2) Subadditivity:

$\forall X_1, X_2 \in M$ it holds $\rho(X_1 + X_2) \le \rho(X_1) + \rho(X_2)$.

(C3) Positive homogeneity:

$\rho(\lambda X) = \lambda \rho(X)$, $\forall \lambda \ge 0$, $\forall X \in M$.

Coherent risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a sample space Ω, a σ-algebra of events $\mathcal{F}$ and a probability measure P.

Let $L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of almost surely finite random variables in $(\Omega, \mathcal{F})$. Let $M \subseteq L^{(0)}$ and $\rho\colon M \to \mathbb{R}$ a risk measure in M.

Definition: A risk measure ρ in M is called *coherent* iff it has the following properties

(C1) Invariance with respect to translation:
$\rho(X + r) = \rho(X) + r$, $\forall r \in \mathbb{R}$ and $\forall X \in M$.

(C2) Subadditivity:
$\forall X_1, X_2 \in M$ it holds $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

(C3) Positive homogeneity:
$\rho(\lambda X) = \lambda \rho(X)$, $\forall \lambda \geq 0$, $\forall X \in M$.

(C4) Monotonicity:
$\forall X_1, X_2 \in M$ the implication $X_1 \overset{a.s.}{\leq} X_2 \Longrightarrow \rho(X_1) \leq \rho(X_2)$ holds.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0, 1]$

$\rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda \rho(X_1) + (1 - \lambda) \rho(X_2)$ holds.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$

$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$ holds.

(C5) is weaker than (C2) and (C3), i.e. (C2) and (C3) together imply (C5), but not vice versa.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$

$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$ holds.

(C5) is weaker than (C2) and (C3), i.e. (C2) and (C3) together imply (C5), but not vice versa.

Definition: A risk measure ρ in M with the properties (C1),(C4) and (C5) is called *convex* in M.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$

$\rho(\lambda X_1 + (1-\lambda) X_2) \leq \lambda \rho(X_1) + (1-\lambda)\rho(X_2)$ holds.

(C5) is weaker than (C2) and (C3), i.e. (C2) and (C3) together imply (C5), but not vice versa.

Definition: A risk measure ρ in M with the properties (C1),(C4) and (C5) is called *convex* in M.

Observation: VaR is not coherent in general.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$

$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$ holds.

(C5) is weaker than (C2) and (C3), i.e. (C2) and (C3) together imply (C5), but not vice versa.

Definition: A risk measure ρ in M with the properties (C1),(C4) and (C5) is called *convex* in M.

Observation: VaR is not coherent in general.
Let the probability measure P be defined by some continuous or discrete probabilty distribution F.
$VaR_\alpha(F) = F^{\leftarrow}(\alpha)$ has the properties (C1), (C3) and (C4), whereas the subadditivity (C2) is not fulfilled in general.

Coherent risk measure (contd.)

Coherent risk measure (contd.)

Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

Coherent risk measure (contd.)

Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

Consider a portfolio consisting of 100 bonds, which default independently with probability p. Observe that the VaR of the portfolio loss is larger than 100 times the VaR of the loss of a single bond.

Coherent risk measure (contd.)

Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

Consider a portfolio consisting of 100 bonds, which default independently with probability p. Observe that the VaR of the portfolio loss is larger than 100 times the VaR of the loss of a single bond.

Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $M \subseteq L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of the random variables with a continuous distribution in $(\Omega, \mathcal{F}, P)$. $CVaR_\alpha$ is a coherent risk measure in M, $\forall \alpha \in (0, 1)$.

Coherent risk measure (contd.)

Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

Consider a portfolio consisting of 100 bonds, which default independently with probability p. Observe that the VaR of the portfolio loss is larger than 100 times the VaR of the loss of a single bond.

Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $M \subseteq L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of the random variables with a continuous distribution in $(\Omega, \mathcal{F}, P)$. $CVaR_\alpha$ is a coherent risk measure in M, $\forall \alpha \in (0, 1)$.

Sketch of the proof:

(C1),(C3), (C4) follow from $CVaR_\alpha(F) = \frac{1}{1-\alpha}\int_\alpha^1 Var_p(F)dp$.

Coherent risk measure (contd.)

Example: Let the probability measure P be defined by the binomial distribution $B(p,n)$ for $n \in \mathbb{N}$, $p \in (0,1)$. We show that $VaR_\alpha(B(p,n))$ is not subadditive.

Consider a portfolio consisting of 100 bonds, which default independently with probability p. Observe that the VaR of the portfolio loss is larger than 100 times the VaR of the loss of a single bond.

Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $M \subseteq L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of the random variables with a continuous distribution in $(\Omega, \mathcal{F}, P)$. $CVaR_\alpha$ is a coherent risk measure in M, $\forall \alpha \in (0,1)$.

Sketch of the proof:

(C1),(C3), (C4) follow from $CVaR_\alpha(F) = \frac{1}{1-\alpha}\int_\alpha^1 Var_p(F)dp$.

To show (C2) observe that for a sequence of i.i.d. r.v. L_1, L_2, ..., L_n with order statistics $L_{1,n} \geq L_{2,n} \geq \ldots \geq L_{n,n}$ and for any $m \in \{1, 2, \ldots, n\}$

$$\sum_{i=1}^{m} L_{i,n} = \sup\{L_{i_1} + L_{i_2} + \ldots + L_{i_m} : 1 \leq i_1 < \ldots < i_m \leq n\} \text{ holds.}$$

The mean-risk portfolio optimization model

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \ldots, X_d)^T$ of their returns. Let $E(X) = \mu$.

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \dots, X_d)^T$ of their returns. Let $E(X) = \mu$.

Let $\mathcal{P}$ be the family of all portfolios consisting of the above d assets

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \ldots, X_d)^T$ of their returns. Let $E(X) = \mu$.

Let $\mathcal{P}$ be the family of all portfolios consisting of the above d assets
Any (long-short) portfolio in $\mathcal{P}$ is uniquelly determined by its weight vector $w = (w_i) \in \mathbb{R}^d$ with $\sum_{i=1^d} |w_i| = 1$. $w_i > 0$ ($w_i < 0$) represents a long (short) investment.

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \ldots, X_d)^T$ of their returns. Let $E(X) = \mu$.

Let $\mathcal{P}$ be the family of all portfolios consisting of the above d assets Any (long-short) portfolio in $\mathcal{P}$ is uniquelly determined by its weight vector $w = (w_i) \in \mathbb{R}^d$ with $\sum_{i=1^d} |w_i| = 1$. $w_i > 0$ ($w_i < 0$) represents a long (short) investment.

The return of portfolio w is the r.v. $Z(w) = \sum_{i=1}^d w_i X_i$. The expected portfolio return is $E(Z(w)) = w^T \mu$.

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \ldots, X_d)^T$ of their returns. Let $E(X) = \mu$.

Let $\mathcal{P}$ be the family of all portfolios consisting of the above d assets
Any (long-short) portfolio in $\mathcal{P}$ is uniquelly determined by its weight vector $w = (w_i) \in \mathbb{R}^d$ with $\sum_{i=1^d} |w_i| = 1$. $w_i > 0$ ($w_i < 0$) represents a long (short) investment.

The return of portfolio w is the r.v. $Z(w) = \sum_{i=1}^d w_i X_i$. The expected portfolio return is $E(Z(w)) = w^T \mu$.

Let $\mathcal{P}_m$ be the family of portfolios in $\mathcal{P}$ with $E(Z(w)) = m$, for some $m \in \mathbb{R}$, $m > 0$.
$\mathcal{P}_m := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1, w^T \mu = m\}$

The mean-risk portfolio optimization model

Consider a portfolio of d risky assets and the random vector $X = (X_1, X_2, \ldots, X_d)^T$ of their returns. Let $E(X) = \mu$.

Let $\mathcal{P}$ be the family of all portfolios consisting of the above d assets
Any (long-short) portfolio in $\mathcal{P}$ is uniquelly determined by its weight vector $w = (w_i) \in \mathbb{R}^d$ with $\sum_{i=1^d} |w_i| = 1$. $w_i > 0$ ($w_i < 0$) represents a long (short) investment.

The return of portfolio w is the r.v. $Z(w) = \sum_{i=1}^d w_i X_i$. The expected portfolio return is $E(Z(w)) = w^T \mu$.

Let $\mathcal{P}_m$ be the family of portfolios in $\mathcal{P}$ with $E(Z(w)) = m$, for some $m \in \mathbb{R}$, $m > 0$.
$\mathcal{P}_m := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1, w^T \mu = m\}$

For a risk measure ρ the **mean-ρ portfolio optimization model** is:

$$\min_{w \in \mathcal{P}_m} \rho(Z(w)) \tag{1}$$

The mean-risk portfolio optimization model (contd.)

The mean-risk portfolio optimization model (contd.)

If ρ equals the portfolio variance we get $\min_{w \in \mathcal{P}_m} var(Z(w))$

The mean-risk portfolio optimization model (contd.)

If ρ equals the portfolio variance we get $\min_{w \in \mathcal{P}_m} var(Z(w))$

With $\Sigma := Cov(X)$ and nonnegative weights $w_i \geq 0$, $i \in \overline{1, d}$, (long-only portfolio) we get the **Markovitz portfolio optimization model** (Markowitz 1952):

$$\begin{aligned} \min_{w} \quad & w^T \Sigma w \\ \text{s.t.} \quad & \\ & w^T \mu = m \\ & \textstyle\sum_{i=1}^{d} w_i = 1 \quad w_i \geq 0,\ i \in \overline{1, d} \end{aligned}$$

The mean-risk portfolio optimization model (contd.)

If ρ equals the portfolio variance we get $\min_{w \in \mathcal{P}_m} var(Z(w))$

With $\Sigma := Cov(X)$ and nonnegative weights $w_i \geq 0$, $i \in \overline{1, d}$, (long-only portfolio) we get the **Markovitz portfolio optimization model** (Markowitz 1952):

$$\begin{aligned} \min_{w} \quad & w^T \Sigma w \\ \text{s.t.} \quad & \\ & w^T \mu = m \\ & \textstyle\sum_{i=1}^{d} w_i = 1 \quad w_i \geq 0,\ i \in \overline{1, d} \end{aligned}$$

If $\rho = VaR_\alpha$, $\alpha \in (0, 1)$ we get the **mean-VaR pf. optimization model**

$$\min_{w \in \mathcal{P}_m} VaR_\alpha(Z(w)).$$

Question: What is the relationship between these three portfolio optimization models?

Answer: In general the three models yield different optimal portfolios!

Mean-risk portfolio optimization in the case of elliptically distributed asset returns

Mean-risk portfolio optimization in the case of elliptically distributed asset returns

Theorem: (Embrechts et al., 2002)
Let M be the set of returns of the portfolii in $\mathcal{P} := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1\}$. Let the asset returns $X = (X_1, X_2, \ldots, X_d)$ be elliptically distributed, $X = (X_1, X_2, \ldots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$. Then VaR_α ist coherent in M, for any $\alpha \in (0.5, 1)$.

Mean-risk portfolio optimization in the case of elliptically distributed asset returns

Theorem: (Embrechts et al., 2002)
Let M be the set of returns of the portfolii in $\mathcal{P} := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1\}$. Let the asset returns $X = (X_1, X_2, \ldots, X_d)$ be elliptically distributed, $X = (X_1, X_2, \ldots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$. Then VaR_α ist coherent in M, for any $\alpha \in (0.5, 1)$.

Theorem: (Embrechts et al., 2002)
Let $X = (X_1, X_2, \ldots, X_d) = \mu + AY$ be elliptically distributed with $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times k}$ and a spherically distributed vector $Y \sim S_k(\psi)$. Assume that $0 < E(X_k^2) < \infty$ holds $\forall k$. If the risk measure ρ has the properties (C1) and (C3) and $\rho(Y_1) > 0$ for the first component Y_1 of Y, then

$$\arg\min\{\rho(Z(w)) \colon w \in \mathcal{P}_m\} = \arg\min\{var(Z(w)) \colon w \in \mathcal{P}_m\}$$

Copulas: Definition and basic properties

Copulas: Definition and basic properties

Definition: A d-dimensional copula is a distribution function on $[0,1]^d$ with uniform marginal distributions on $[0,1]$.

Copulas: Definition and basic properties

Definition: A d-dimensional copula is a distribution function on $[0,1]^d$ with uniform marginal distributions on $[0,1]$.

Equivalently, a copula C is a function $C\colon [0,1]^d \to [0,1]$, with the following properties:

1. $C(u_1, u_2, \ldots, u_d)$ is mon. increasing in each variable u_i, $1 \le i \le d$.
2. $C(1, 1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$ for any $k \in \{1, \ldots, d\}$ and $\forall u_k \in [0,1]$.
3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1, 2, \ldots, d\}$:

$$\sum_{k_1=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$.

Copulas: Definition and basic properties

Definition: A d-dimensional copula is a distribution function on $[0,1]^d$ with uniform marginal distributions on $[0,1]$.

Equivalently, a copula C is a function $C\colon [0,1]^d \to [0,1]$, with the following properties:

1. $C(u_1, u_2, \ldots, u_d)$ is mon. increasing in each variable u_i, $1 \le i \le d$.
2. $C(1, 1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$ for any $k \in \{1, \ldots, d\}$ and $\forall u_k \in [0,1]$.
3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1, 2, \ldots, d\}$:

$$\sum_{k_1=1}^{2} \ldots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$.

Remark: The k-dimensional marginal distributions of a d-dimensional copula are k-dimensional copulas, for all $2 \le k \le d$.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff$ $h^{\leftarrow}$ is continuous.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff$ $h^{\leftarrow}$ is continuous.
4. $h^{\leftarrow}(h(x)) \leq x$

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff$ $h^{\leftarrow}$ is continuous.
4. $h^{\leftarrow}(h(x)) \leq x$
5. $h(h^{\leftarrow}(y)) \geq y$

Lemma: Let $h \colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow} \colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff$ $h^{\leftarrow}$ is continuous.
4. $h^{\leftarrow}(h(x)) \le x$
5. $h(h^{\leftarrow}(y)) \ge y$
6. h is strictly monotone increasing $\implies$ $h^{\leftarrow}(h(x)) = x$.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff h^{\leftarrow}$ is continuous.
4. $h^{\leftarrow}(h(x)) \leq x$
5. $h(h^{\leftarrow}(y)) \geq y$
6. h is strictly monotone increasing $\Longrightarrow h^{\leftarrow}(h(x)) = x$.
7. h is continuous $\Longrightarrow h(h^{\leftarrow}(y)) = y$.

Lemma: Let $h \colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow} \colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is a monotone increasing left continuous function.
2. h is continuous $\iff$ $h^{\leftarrow}$ is strictly monotone increasing.
3. h is strictly monotone increasing $\iff$ $h^{\leftarrow}$ is continuous.
4. $h^{\leftarrow}(h(x)) \le x$
5. $h(h^{\leftarrow}(y)) \ge y$
6. h is strictly monotone increasing $\Longrightarrow$ $h^{\leftarrow}(h(x)) = x$.
7. h is continuous $\Longrightarrow$ $h(h^{\leftarrow}(y)) = y$.

Lemma: Let X be a r.v. with continuous distribution function F. Then $\mathbb{P}\left(F^{\leftarrow}(F(x)) = x\right) = 1$, i.e. $F^{\leftarrow}(F(X)) \overset{a.s.}{=} X$

Copulas: existence and uniqueness

Copulas: existence and uniqueness

Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
   If $U \sim U(0,1)$, then $\mathbb{P}(G^{\leftarrow}(U) \leq x) = G(x)$.

Copulas: existence and uniqueness

Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
   If $U \sim U(0,1)$, then $\mathbb{P}(G^{\leftarrow}(U) \leq x) = G(x)$.
2. Probability transformation: Let Y be a r.v. with continuous d.f. G.
   Then $G(Y) \sim U(0,1)$.

Copulas: existence and uniqueness

Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
   If $U \sim U(0,1)$, then $\mathbb{P}(G^{\leftarrow}(U) \leq x) = G(x)$.
2. Probability transformation: Let Y be a r.v. with continuous d.f. G.
   Then $G(Y) \sim U(0,1)$.

Theorem: (Sklar, 1959)
Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. F_1,...,F_d. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

If F_1,...,F_d are continuous, then C is unique.

Copulas: existence and uniqueness

Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
   If $U \sim U(0,1)$, then $\mathbb{P}(G^{\leftarrow}(U) \leq x) = G(x)$.
2. Probability transformation: Let Y be a r.v. with continuous d.f. G.
   Then $G(Y) \sim U(0,1)$.

Theorem: (Sklar, 1959)
Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. F_1,...,F_d. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

If F_1,...,F_d are continuous, then C is unique.
Vice-versa, if C is a copula and F_1,...,F_d are d.f., then the function F defined by the equality above is a joint d.f. with marginal d.f. F_1,...,F_d.

Copulas: existence and uniqueness

Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
   If $U \sim U(0,1)$, then $\mathbb{P}(G^{\leftarrow}(U) \leq x) = G(x)$.
2. Probability transformation: Let Y be a r.v. with continuous d.f. G.
   Then $G(Y) \sim U(0,1)$.

Theorem: (Sklar, 1959)
Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. F_1,. . .,F_d. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

If F_1,. . .,F_d are continuous, then C is unique.
Vice-versa, if C is a copula and F_1,. . .,F_d are d.f., then the function F defined by the equality above is a joint d.f. with marginal d.f. F_1,. . .,F_d.

C as above is called *the copula of* F. For a random vector $X \in \mathbb{R}^d$ with c.d.f. F we say that C is *the copula of* X.

Copulas: invariance property

Copulas: invariance property

Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1,\ldots,F_d$. The unique copula C of F is given as :

$$C(u_1, u_2, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2), \ldots, F_d^{\leftarrow}(u_d)).$$

Copulas: invariance property

Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1,\ldots,F_d$. The unique copula C of F is given as :

$$C(u_1, u_2, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2), \ldots, F_d^{\leftarrow}(u_d)).$$

Theorem: (Copula invariance w.r.t. strictly monotone transformations)
Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let $T_1, T_2, \ldots, T_d$ be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

Copulas: invariance property

Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1,\ldots,F_d$. The unique copula C of F is given as :

$$C(u_1, u_2, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2), \ldots, F_d^{\leftarrow}(u_d)).$$

Theorem: (Copula invariance w.r.t. strictly monotone transformations)
Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let $T_1, T_2, \ldots, T_d$ be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

Example: Let $X = (X_1, \ldots, X_d) \sim N_d(0, \Sigma)$ with $\Sigma = R$ being the correlation matrix of X. Let ϕ_R and ϕ be the c.d.f of X and X_1, resp.. The copula of X is called **a Gaussian copula** and is denoted by C_R^{Ga}:

$$C_R^{Ga}(u_1, u_2, \ldots, u_d) = \phi_R(\phi^{-1}(u_1), \phi^{-1}(u_2), \ldots, \phi^{-1}(u_d)).$$

Copulas: invariance property

Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1,\ldots,F_d$. The unique copula C of F is given as :

$$C(u_1, u_2, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2), \ldots, F_d^{\leftarrow}(u_d)).$$

Theorem: (Copula invariance w.r.t. strictly monotone transformations)

Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let $T_1, T_2, \ldots, T_d$ be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

Example: Let $X = (X_1, \ldots, X_d) \sim N_d(0, \Sigma)$ with $\Sigma = R$ being the correlation matrix of X. Let ϕ_R and ϕ be the c.d.f of X and X_1, resp.. The copula of X is called **a Gaussian copula** and is denoted by C_R^{Ga}:

$$C_R^{Ga}(u_1, u_2, \ldots, u_d) = \phi_R(\phi^{-1}(u_1), \phi^{-1}(u_2), \ldots, \phi^{-1}(u_d)).$$

C_R^{Ga} is the copula of any non-degenerate normal distribution $N_d(\mu, \Sigma)$ with correlation matrix R.

Copulas: invariance property

Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1,\ldots,F_d$. The unique copula C of F is given as :

$$C(u_1, u_2, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), F_2^{\leftarrow}(u_2), \ldots, F_d^{\leftarrow}(u_d)).$$

Theorem: (Copula invariance w.r.t. strictly monotone transformations) Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let $T_1, T_2, \ldots, T_d$ be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

Example: Let $X = (X_1, \ldots, X_d) \sim N_d(0, \Sigma)$ with $\Sigma = R$ being the correlation matrix of X. Let ϕ_R and ϕ be the c.d.f of X and X_1, resp.. The copula of X is called **a Gaussian copula** and is denoted by C_R^{Ga}:

$$C_R^{Ga}(u_1, u_2, \ldots, u_d) = \phi_R(\phi^{-1}(u_1), \phi^{-1}(u_2), \ldots, \phi^{-1}(u_d)).$$

C_R^{Ga} is the copula of any non-degenerate normal distribution $N_d(\mu, \Sigma)$ with correlation matrix R.

For $d = 2$ and $\rho = R_{12} \in (-1, 1)$ we have :

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)}\right\} dx_1 dx_2$$

Copulas: lower and upper bounds

Copulas: lower and upper bounds

Theorem: (Fréchet bounds)
The following inequalities hold for any d-dimensional copula C and any $(u_1, u_2, \ldots, u_d) \in [0,1]^d$, where $d \in \mathbb{N}$:

$$\max\left\{\sum_{k=1}^{d} u_k - d + 1, 0\right\} \leq C(u_1, u_2, \ldots, u_d) \leq \min\{u_1, u_2, \ldots, u_d\}.$$

Copulas: lower and upper bounds

Theorem: (Fréchet bounds)
The following inequalities hold for any d-dimensional copula C and any $(u_1, u_2, \ldots, u_d) \in [0,1]^d$, where $d \in \mathbb{N}$:

$$\max\left\{\sum_{k=1}^{d} u_k - d + 1, 0\right\} \le C(u_1, u_2, \ldots, u_d) \le \min\{u_1, u_2, \ldots, u_d\}.$$

Notation: Lower bound =: W_d, upper bound =: M_d, for $d \ge 2$.
For $d = 2$ we write $M := M_2$, $W := W_2$.

Copulas: lower and upper bounds

Theorem: (Fréchet bounds)
The following inequalities hold for any d-dimensional copula C and any $(u_1, u_2, \ldots, u_d) \in [0,1]^d$, where $d \in \mathbb{N}$:

$$\max\left\{\sum_{k=1}^{d} u_k - d + 1, 0\right\} \leq C(u_1, u_2, \ldots, u_d) \leq \min\{u_1, u_2, \ldots, u_d\}.$$

Notation: Lower bound =: W_d, upper bound =: M_d, for $d \geq 2$.
For $d = 2$ we write $M := M_2$, $W := W_2$.

Remark: Analogous inequalities hold for any general c.d.f. F with marginal d.f. F_i, $1 \leq i \leq d$:

$$\max\left\{\sum_{k=1}^{d} F_k(x_k) - d + 1, 0\right\} \leq F(x_1, x_2, \ldots, x_d) \leq \min\{F_1(x_1), F_2(x_2), \ldots, F_d(x_d)\}.$$

Copulas: lower and upper bounds

Theorem: (Fréchet bounds)
The following inequalities hold for any d-dimensional copula C and any $(u_1, u_2, \ldots, u_d) \in [0,1]^d$, where $d \in \mathbb{N}$:

$$\max\left\{\sum_{k=1}^{d} u_k - d + 1, 0\right\} \le C(u_1, u_2, \ldots, u_d) \le \min\{u_1, u_2, \ldots, u_d\}.$$

Notation: Lower bound $=: W_d$, upper bound $=: M_d$, for $d \geq 2$.
For $d = 2$ we write $M := M_2$, $W := W_2$.

Remark: Analogous inequalities hold for any general c.d.f. F with marginal d.f. F_i, $1 \le i \le d$:

$$\max\left\{\sum_{k=1}^{d} F_k(x_k) - d + 1, 0\right\} \le F(x_1, x_2, \ldots, x_d) \le \min\{F_1(x_1), F_2(x_2), \ldots, F_d(x_d)\}.$$

Exercise: The Fréchet lower bound W_d is not a copula for $d \geq 3$.

Hint: Check that the rectangle inequality $\sum_{k_1=1}^{2}\sum_{k_2=1}^{2}\ldots\sum_{k_d=1}^{2}(-1)^{k_1+k_2+\ldots+k_d} W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0$ with $u_{j1} = a_j$ and $u_{j2} = b_j$ for $j \in \{1, 2, \ldots, d\}$, is not fulfilled for $d \geq 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.

Copulas: lower and upper bounds (contd.)

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Remark 1: The Fréchet upper bound M_d is a copula for any $d \in \mathbb{N}$, $d \geq 2$.

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Remark 1: The Fréchet upper bound M_d is a copula for any $d \in \mathbb{N}$, $d \geq 2$.
The fulfillment of the three copula axioms is simple to check.

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Remark 1: The Fréchet upper bound M_d is a copula for any $d \in \mathbb{N}$, $d \geq 2$.
The fulfillment of the three copula axioms is simple to check.

Remark 2: M and W are copulas.

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Remark 1: The Fréchet upper bound M_d is a copula for any $d \in \mathbb{N}$, $d \geq 2$.
The fulfillment of the three copula axioms is simple to check.

Remark 2: M and W are copulas.
Hint: Let X be a r.v. eine with d.f. F_X, let T be a strictly monotone increasing function, and let S be a strictly monotone decreasing function. Consider the r.v. $Y := T(X)$ and $Z := S(X)$.

Copulas: lower and upper bounds (contd.)

Theorem: (for a proof see Nelsen 1999)
For any $d \in \mathbb{N}$, $d \geq 3$, and any $u \in [0,1]^d$, there exists a copula $C_{d,u}$ such that $C_{d,u}(u) = W_d(u)$.

Remark 1: The Fréchet upper bound M_d is a copula for any $d \in \mathbb{N}$, $d \geq 2$.
The fulfillment of the three copula axioms is simple to check.

Remark 2: M and W are copulas.
Hint: Let X be a r.v. eine with d.f. F_X, let T be a strictly monotone increasing function, and let S be a strictly monotone decreasing function. Consider the r.v. $Y := T(X)$ and $Z := S(X)$.

Then M is the copula of $(X, T(X))^T$ and W is the copula of $(X, S(X))^T$.

Copulas: co-monotonicity and anti-monotonicity

Copulas: co-monotonicity and anti-monotonicity

Definition: X_1 and X_2 are called co-monotone if M is a copula of $(X_1, X_2)^T$. X_1 snd X_2 are called anti-monotone if W is a copula of $(X_1, X_2)^T$.

Copulas: co-monotonicity and anti-monotonicity

Definition: X_1 and X_2 are called co-monotone if M is a copula of $(X_1, X_2)^T$. X_1 snd X_2 are called anti-monotone if W is a copula of $(X_1, X_2)^T$.

Theorem: Assume that W or M is a copula of $(X_1, X_2)^T$. Then there exist two monotone functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ and a r.v. Z, such that

$$(X_1, X_2) \stackrel{d}{=} (\alpha(Z), \beta(Z)) \, .$$

If M is the copula of $(X_1, X_2)^T$, then both α and β are monotone increasing, if W is the copula of $(X_1, X_2)^T$, then one of the functions α, β is monotone increasing and the other one is monotone decreasing.

Copulas: co-monotonicity and anti-monotonicity

Definition: X_1 and X_2 are called co-monotone if M is a copula of $(X_1, X_2)^T$. X_1 snd X_2 are called anti-monotone if W is a copula of $(X_1, X_2)^T$.

Theorem: Assume that W or M is a copula of $(X_1, X_2)^T$. Then there exist two monotone functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ and a r.v. Z, such that

$$(X_1, X_2) \stackrel{d}{=} (\alpha(Z), \beta(Z)).$$

If M is the copula of $(X_1, X_2)^T$, then both α and β are monotone increasing, if W is the copula of $(X_1, X_2)^T$, then one of the functions α, β is monotone increasing and the other one is monotone decreasing.

If C is the copula of (X_1, X_2) and the marginal d.f. F_1 and F_2 of (X_1, X_2) are continuous, then the following hold:

$C = W$ iff $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$ monotone decreasing,
$C = M$ iff $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$ monotone increasing.

Copulas: co-monotonicity and anti-monotonicity

Definition: X_1 and X_2 are called co-monotone if M is a copula of $(X_1, X_2)^T$. X_1 snd X_2 are called anti-monotone if W is a copula of $(X_1, X_2)^T$.

Theorem: Assume that W or M is a copula of $(X_1, X_2)^T$. Then there exist two monotone functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ and a r.v. Z, such that

$$(X_1, X_2) \stackrel{d}{=} (\alpha(Z), \beta(Z)) \, .$$

If M is the copula of $(X_1, X_2)^T$, then both α and β are monotone increasing, if W is the copula of $(X_1, X_2)^T$, then one of the functions α, β is monotone increasing and the other one is monotone decreasing.

If C is the copula of (X_1, X_2) and the marginal d.f. F_1 and F_2 of (X_1, X_2) are continuous, then the following hold:

$C = W$ iff $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$ monotone decreasing,
$C = M$ iff $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$ monotone increasing.

Proof: In McNeil et al., 2005.

Copulas: bounds for the linear correlation

Copulas: bounds for the linear correlation

Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

Copulas: bounds for the linear correlation

Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

1. The possible values of the linear correlation coefficient of X_1 and X_2 build a closed interval $[\rho_{L,min}; \rho_{L,max}]$ with $0 \in [\rho_{L,min}; \rho_{L,max}]$.

Copulas: bounds for the linear correlation

Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

1. The possible values of the linear correlation coefficient of X_1 and X_2 build a closed interval $[\rho_{L,min}; \rho_{L,max}]$ with $0 \in [\rho_{L,min}; \rho_{L,max}]$.
2. The minimal linear correlation $\rho_{L,min}$ is reached iff X_1 and X_2 are anti-monotone. The maximal linear correlation $\rho_{L,max}$ is reached iff X_1 and X_2 are co-monotone.

Copulas: bounds for the linear correlation

Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

1. The possible values of the linear correlation coefficient of X_1 and X_2 build a closed interval $[\rho_{L,min}; \rho_{L,max}]$ with $0 \in [\rho_{L,min}; \rho_{L,max}]$.
2. The minimal linear correlation $\rho_{L,min}$ is reached iff X_1 and X_2 are anti-monotone. The maximal linear correlation $\rho_{L,max}$ is reached iff X_1 and X_2 are co-monotone.

The proof uses the equality of Höffding:
Lemma: (The Höffding equality)
Let $(X_1, X_2)^T$ be a random vector with c.d.f. F and marginal d.f. F_1, F_2. If $cov(X_1, X_2) < \infty$ then the following equality holds:

$$cov(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2))dx_1 dx_2 .$$

Copulas: bounds for the linear correlation

Theorem: Let $(X_1, X_2)^T$ be a random vector with marginal d.f. F_1, F_2 and some unknown copula. Let $var(X_1), var(X_2) \in (0, \infty)$ hold. Then the following statements hold:

1. The possible values of the linear correlation coefficient of X_1 and X_2 build a closed interval $[\rho_{L,min}; \rho_{L,max}]$ with $0 \in [\rho_{L,min}; \rho_{L,max}]$.
2. The minimal linear correlation $\rho_{L,min}$ is reached iff X_1 and X_2 are anti-monotone. The maximal linear correlation $\rho_{L,max}$ is reached iff X_1 and X_2 are co-monotone.

The proof uses the equality of Höffding:
Lemma: (The Höffding equality)
Let $(X_1, X_2)^T$ be a random vector with c.d.f. F and marginal d.f. F_1, F_2. If $cov(X_1, X_2) < \infty$ then the following equality holds:

$$cov(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2)) dx_1 dx_2 \,.$$

Proof in McNeil et al., 2005.

Copulas: bounds for the linear correlation (examples)

Copulas: bounds for the linear correlation (examples)

Example: Let X_1, X_2 be two random variables with $X_1 \sim \mathit{Lognormal}(0,1)$, $X_2 \sim \mathit{Lognormal}(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

Copulas: bounds for the linear correlation (examples)

Example: Let X_1, X_2 be two random variables with $X_1 \sim \mathit{Lognormal}(0,1)$, $X_2 \sim \mathit{Lognormal}(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{-\sigma Z}$ are anti-monotone.

Copulas: bounds for the linear correlation (examples)

Example: Let X_1, X_2 be two random variables with $X_1 \sim \mathit{Lognormal}(0,1)$, $X_2 \sim \mathit{Lognormal}(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{-\sigma Z}$ are anti-monotone.

Example: Determine two random vectors $(X_1,X_2)^T$ and $(Y_1,Y_2)^T$ with different c.d.f.s such that $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds while $X_1, X_2, Y_1, Y_2 \sim N(0,1)$ and $\rho_L(X_1,X_2) = 0$, $\rho_L(Y_1,Y_2) = 0$ also hold.

Copulas: bounds for the linear correlation (examples)

Example: Let X_1, X_2 be two random variables with $X_1 \sim Lognormal(0,1)$, $X_2 \sim Lognormal(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{-\sigma Z}$ are anti-monotone.

Example: Determine two random vectors $(X_1,X_2)^T$ and $(Y_1,Y_2)^T$ with different c.d.f.s such that $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds while $X_1, X_2, Y_1, Y_2 \sim N(0,1)$ and $\rho_L(X_1,X_2) = 0$, $\rho_L(Y_1,Y_2) = 0$ also hold.

If $(X_1,X_2)^T$, $(Y_1,Y_2)^T$ represent the asset returns of two different portfolios consisting of two assets each, then we have two portfolios with the same marginal distributions of their assets and the same linear correlation coefficient, respectively, but having different value at risk.

Copulas: bounds for the linear correlation (examples)

Example: Let X_1, X_2 be two random variables with $X_1 \sim Lognormal(0,1)$, $X_2 \sim Lognormal(0,\sigma^2)$, $\sigma > 0$. Determine Sie $\rho_{L,min}(X_1,X_2)$ und $\rho_{L,max}(X_1,X_2)$.

Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{-\sigma Z}$ are anti-monotone.

Example: Determine two random vectors $(X_1,X_2)^T$ and $(Y_1,Y_2)^T$ with different c.d.f.s such that $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds while $X_1, X_2, Y_1, Y_2 \sim N(0,1)$ and $\rho_L(X_1,X_2) = 0$, $\rho_L(Y_1,Y_2) = 0$ also hold.

If $(X_1,X_2)^T$, $(Y_1,Y_2)^T$ represent the asset returns of two different portfolios consisting of two assets each, then we have two portfolios with the same marginal distributions of their assets and the same linear correlation coefficient, respectively, but having different value at risk.

Conclusion: The marginal distributions of the assets in a portfolio and the linear correlation between the assets do not determine the loss distribution, in particular, they do not determine the risk measure of the portfolio.

The rang correlation Kendall's Tau

The rang correlation Kendall's Tau

Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two samples of a random vector $(X, Y)^T$. $(x, y)^T$ und $(\tilde{x}, \tilde{y})^T$ are called *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$ and *discordant* if $(x - \tilde{x})(y - \tilde{y}) < 0$.

The rang correlation Kendall's Tau

Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two samples of a random vector $(X, Y)^T$. $(x, y)^T$ und $(\tilde{x}, \tilde{y})^T$ are called *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$ and *discordant* if $(x - \tilde{x})(y - \tilde{y}) < 0$.

Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Kendall's Tau of $(X_1, X_2)^T$ is defined as $\rho_\tau(X_1, X_2) = \mathbb{P}((X_1 - X_1')(X_2 - X_2') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2') < 0)$, where $(X_1', X_2')^T$ is an independent copy of $(X_1, X_2)^T$.

The rang correlation Kendall's Tau

Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two samples of a random vector $(X, Y)^T$. $(x, y)^T$ und $(\tilde{x}, \tilde{y})^T$ are called *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$ and *discordant* if $(x - \tilde{x})(y - \tilde{y}) < 0$.

Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Kendall's Tau of $(X_1, X_2)^T$ is defined as $\rho_\tau(X_1, X_2) = \mathbb{P}((X_1 - X_1')(X_2 - X_2') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2') < 0)$, where $(X_1', X_2')^T$ is an independent copy of $(X_1, X_2)^T$.

Equivalently: $\rho_\tau(X_1, X_2) = E(sign[(X_1 - X_1')(X_2 - X_2')])$.

The rang correlation Kendall's Tau

Let $(x,y)^T$ and $(\tilde{x},\tilde{y})^T$ be two samples of a random vector $(X,Y)^T$. $(x,y)^T$ und $(\tilde{x},\tilde{y})^T$ are called *concordant* if $(x-\tilde{x})(y-\tilde{y}) > 0$ and *discordant* if $(x-\tilde{x})(y-\tilde{y}) < 0$.

Definition: Let $(X_1,X_2)^T$ be a random vector with continuous marginal distributions. The Kendall's Tau of $(X_1,X_2)^T$ is defined as $\rho_\tau(X_1,X_2) = \mathbb{P}((X_1-X_1')(X_2-X_2') > 0) - \mathbb{P}((X_1-X_1')(X_2-X_2') < 0)$, where $(X_1',X_2')^T$ is an independent copy of $(X_1,X_2)^T$.

Equivalently: $\rho_\tau(X_1,X_2) = E(sign[(X_1-X_1')(X_2-X_2')])$. In the d-dimensional case $X \in \mathbb{R}^d$: $\rho_\tau(X) = cov(sign(X-X'))$, where $X' \in \mathbb{R}^D$ is an independent copy of $X \in \mathbb{R}^d$.

The rang correlation Kendall's Tau

Let $(x,y)^T$ and $(\tilde{x},\tilde{y})^T$ be two samples of a random vector $(X,Y)^T$. $(x,y)^T$ und $(\tilde{x},\tilde{y})^T$ are called *concordant* if $(x-\tilde{x})(y-\tilde{y})>0$ and *discordant* if $(x-\tilde{x})(y-\tilde{y})<0$.

Definition: Let $(X_1,X_2)^T$ be a random vector with continuous marginal distributions. The Kendall's Tau of $(X_1,X_2)^T$ is defined as $\rho_\tau(X_1,X_2)=\mathbb{P}((X_1-X_1')(X_2-X_2')>0)-\mathbb{P}((X_1-X_1')(X_2-X_2')<0)$, where $(X_1',X_2')^T$ is an independent copy of $(X_1,X_2)^T$.

Equivalently: $\rho_\tau(X_1,X_2)=E(sign[(X_1-X_1')(X_2-X_2')])$. In the d-dimensional case $X\in\mathbb{R}^d$: $\rho_\tau(X)=cov(sign(X-X'))$, where $X'\in\mathbb{R}^D$ is an independent copy of $X\in\mathbb{R}^d$.

The sample Kendall's Tau:

Let $\{(x_1,y_1)^T,(x_2,y_2)^T,\ldots,(x_n,y_n)^T\}$ be a sample of size n of the random vector $(X,Y)^T$ with continuous marginal distributions. Let c be the number concordant pairs in the sample and let d be the number of discordant pairs. Then the sample Kendall's Tau is given as

$$\tilde{\rho}_\tau(X,Y)=\frac{c-d}{c+d}\overset{a.s.}{=}\frac{c-d}{n(n-1)/2}$$

The rang correlation Spearman's Rho

The rang correlation Spearman's Rho

Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Spearman's Rho of $(X_1, X_2)^T$ is defined as:

$$\rho_S(X_1, X_2) = 3(\mathbb{P}((X_1 - X_1')(X_2 - X_2'') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2'') < 0)),$$

where $(X_1', X_2')^T$, $(X_1'', X_2'')^T$ are i.i.d. copies of $(X_1, X_2)^T$.

The rang correlation Spearman's Rho

Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Spearman's Rho of $(X_1, X_2)^T$ is defined as:

$$\rho_S(X_1, X_2) = 3(\mathbb{P}((X_1 - X_1')(X_2 - X_2'') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2'') < 0)),$$

where $(X_1', X_2')^T$, $(X_1'', X_2'')^T$ are i.i.d. copies of $(X_1, X_2)^T$.

Equivalent definition (without a proof):
Let F_1 und F_2 be the continuous marginal distributions of $(X_1, X_2)^T$.
Then $\rho_S(X_1, X_2) = \rho_L(F_1(X_1), F_2(X_2))$ holds, i.e. the Spearman's Rho is the linear correlation of the unique copula of $(X_1, X_2)^T$.

The rang correlation Spearman's Rho

Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Spearman's Rho of $(X_1, X_2)^T$ is defined as:

$$\rho_S(X_1, X_2) = 3(\mathbb{P}((X_1 - X_1')(X_2 - X_2'') > 0) - \mathbb{P}((X_1 - X_1')(X_2 - X_2'') < 0)),$$

where $(X_1', X_2')^T$, $(X_1'', X_2'')^T$ are i.i.d. copies of $(X_1, X_2)^T$.

Equivalent definition (without a proof):

Let F_1 und F_2 be the continuous marginal distributions of $(X_1, X_2)^T$. Then $\rho_S(X_1, X_2) = \rho_L(F_1(X_1), F_2(X_2))$ holds, i.e. the Spearman's Rho is the linear correlation of the unique copula of $(X_1, X_2)^T$.

In the d-dimensional case $X \in \mathbb{R}^d$:

$\rho_S(X) = \rho(F_1(X_1), F_2(X_2), \ldots, F_d(X_d))$ is the correlation matrix of the unique copula of X, where F_1,F_2,. . .,F_d are the continuous marginal distributions of X.

Properties of ρ_τ and ρ_S.

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$$

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$$

$$\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2 =$$

$$12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3$$

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1$$
$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 (C(u_1, u_2) - u_1u_2)du_1du_2 =$$
$$12\int_0^1\int_0^1 C(u_1, u_2)du_1du_2 - 3$$

- ρ_τ and ρ_S are symmetric and take their values on $[-1, 1]$.

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1$$

$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 (C(u_1, u_2) - u_1 u_2)du_1 du_2 =$$
$$12\int_0^1\int_0^1 C(u_1, u_2)du_1 du_2 - 3$$

- ρ_τ and ρ_S are symmetric and take their values on $[-1, 1]$.
- If X_1, X_2 are independent, then $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 0$. In general the converse does not hold.

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1$$
$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 (C(u_1, u_2) - u_1 u_2)du_1 du_2 =$$
$$12\int_0^1\int_0^1 C(u_1, u_2)du_1 du_2 - 3$$

- ρ_τ and ρ_S are symmetric and take their values on $[-1, 1]$.
- If X_1, X_2 are independent, then $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 0$. In general the converse does not hold.
- X_1, X_2 are co-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 1$. X_1, X_2. X_1, X_2 are anti-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = -1$.

Properties of ρ_τ and ρ_S.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and unique copula C. The following equalities hold:

$$\rho_\tau(X_1, X_2) = 4\int_0^1\int_0^1 C(u_1, u_2)dC(u_1, u_2) - 1$$
$$\rho_S(X_1, X_2) = 12\int_0^1\int_0^1 (C(u_1, u_2) - u_1 u_2)du_1 du_2 =$$
$$12\int_0^1\int_0^1 C(u_1, u_2)du_1 du_2 - 3$$

- ρ_τ and ρ_S are symmetric and take their values on $[-1, 1]$.
- If X_1, X_2 are independent, then $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 0$. In general the converse does not hold.
- X_1, X_2 are co-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = 1$. X_1, X_2. X_1, X_2 are anti-monotone iff $\rho_\tau(X_1, X_2) = \rho_S(X_1, X_2) = -1$.
- Let F_1, F_2 be the continuous marginal distributions of $(X_1, X_2)^T$ and let T_1, T_2 be strictly monotone functions on $[-\infty, \infty]$. Then the following equalities hold $\rho_\tau(X_1, X_2) = \rho_\tau(T_1(X_1), T_2(X_2))$ and $\rho_S(X_1, X_2) = \rho_S(T_1(X_1), T_2(X_2))$.

(See Embrechts et al., 2002).