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3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1, 2, \ldots, d\}$:

$$\sum_{k_1=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

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Remark: The k-dimensional marginal distributions of a d-dimensional copula are k-dimensional copulas, for all $2 \le k \le d$.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

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Lemma: Let X be a r.v. with continuous distribution function F. Then $\mathbb{P}\left(F^{\leftarrow}(F(x)) = x\right) = 1$, i.e. $F^{\leftarrow}(F(X)) \overset{a.s.}{=} X$

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Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

1. Quantile transformation:
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Theorem: (Sklar, 1959)
Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. F_1,...,F_d. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

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C as above is called *the copula of* F. For a random vector $X \in \mathbb{R}^d$ with c.d.f. F we say that C is *the copula of* X.

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Corollary: Let F be a c.d.f. with continuous marginal d.f. F_1,. . .,F_d. The unique copula C of F is given as :

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Theorem: (Copula invariance w.r.t. strictly monotone transformations)
Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let T_1,T_2,..., T_d be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

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Example: Let $X = (X_1, \ldots, X_d) \sim N_d(0, \Sigma)$ with $\Sigma = R$ being the correlation matrix of X. Let ϕ_R and ϕ be the c.d.f of X and X_1, resp.. The copula of X is called **a Gaussian copula** and is denoted by C_R^{Ga}:

$$C_R^{Ga}(u_1, u_2, \ldots, u_d) = \phi_R(\phi^{-1}(u_1), \phi^{-1}(u_2), \ldots, \phi^{-1}(u_d)).$$

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For $d = 2$ and $\rho = R_{12} \in (-1, 1)$ we have :

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)}\right\} dx_1 dx_2$$

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Theorem: (Fréchet bounds)
The following inequalities hold for any d-dimensional copula C and any $(u_1, u_2, \ldots, u_d) \in [0,1]^d$, where $d \in \mathbb{N}$:

$$\max\left\{\sum_{k=1}^{d} u_k - d + 1, 0\right\} \leq C(u_1, u_2, \ldots, u_d) \leq \min\{u_1, u_2, \ldots, u_d\}.$$

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Exercise: The Fréchet lower bound W_d is not a copula for $d \geq 3$.

Hint: Check that the rectangle inequality
$\sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \ldots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \geq 0$ with $u_{j1} = a_j$ and $u_{j2} = b_j$ for $j \in \{1, 2, \ldots, d\}$, is not fulfilled for $d \geq 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.