Risk theory and management in actuarial science
winter term 2020/21

2nd work sheet

8. Consider a portfolio consisting of the following index futures *Dow Jones Industrial Average (^DJI), S&P 500 ETF (^GSPC), Nasdaq Composite (^IXIC), DAX (^GDAXI)* and *ATX (^ATX)* with one piece per each index future. Estimate value at risk $\mathrm{VaR}_{0.90}$ of the weekly losses of this portfolio in two ways as described below. Then compare and comment on the obtained results.

   (a) Use historical simulation of the losses over the last 10 years, from October 20, 2008 until October 19, 2020.

   (b) Use the variance-covariance method, again based on the weekly logarithmic returns of the last 10 years as in (a).

   The data can be downloaded from `finance.yahoo.com`: search for the required index future (you can well search for the abbreviations given in paranthesis above), click 'Historical Prices', update the 'Time Period' and 'Frequency' appropriately, and finally klick on 'Download data'. Use the adjusted close prices to compute the weakly logarithmic returns of the single index futures.

9. Show that the following distributions are regularly varying:

   (a) The Pareto distribution G_α with parameter α given as $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$, where $\alpha > 0$. Show that $\bar{G}_\alpha(tx)/\bar{G}_\alpha(t) = x^{-\alpha}$ holds for $t > 0$, $x > 0$, thus $\bar{G}_\alpha \in RV_{-\alpha}$.

   (b) The Fréchet distribution Φ_α with parameter α given as $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, where $\alpha > 0$. Show that $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

10. Let X and Y be positive random variables representing losses in two lines of business (e.g. losses due to fire and car accidents) of an insurance company. Suppose that X has distribution function F which satisfies $\bar{F} \in RV_{-\alpha}$ for $\alpha > 0$. Moreover suppose that Y has finite moments of all orders, i.e. $E(Y^k) < \infty$, for every $k > 0$. Compute $\lim_{x\to\infty} P(X > x | X + Y > x)$, i.e. the asymptotic probability of a large loss in the fire insurance line given a large total loss.

11. Prove the following characterization of the maximum domain of attraction of an extreme value distribution H (also fomulated in the lecture):
    $F \in MDA(H)$ with normalizing and centralizing constants $a_n > 0$, b_n, $n \in \mathbb{N}$, respectively, iff $\lim_{n\to\infty} n\bar{F}_n(a_n x + b) = -\ln(H(x))$, for all $x \in \mathbb{R}$.

12. (The Maxima of the Poisson distribution)
    Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, for some parameter $\lambda > 0$. Show that there exists no extreme value distribution Z such that $X \in MDA(Z)$.

    Hint: Use the following lemma of Leadbetter et al.[1].
    For any discrete non-negative distribution F with right end $x_F = +\infty$ (i.e. a random variable with distribution F can take arbitrarily large values), the following two statements are equivalent for every $\tau \in (0, \infty)$: a) there exists a sequence $u_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} n\bar{F}(u_n) = \tau$, and b) $\lim_{n\to\infty} \frac{\bar{F}(n)}{\bar{F}(n-1)} = 1$.

    You don't need to prove the lemma.

13. (Maximum domain of attraction of the Fréchet distribution)
    Show that the following distributions belong to the maximum domain of attraction $MDA(\Phi_\alpha)$ of the Fréchet distribution Φ_α, for some $\alpha > 0$, and determine the normalizing and centralizing constants $a_n > 0$, b_n, for $n \in \mathbb{N}$, respectively.

[1] M.R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and related properties of random sequences and processes*, Springer, Berlin, 1983.

(a) The Pareto distribution with parameter $\alpha > 0$: $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$.

(b) The Cauchy distribution with density function $f(x) = (\pi(1+x^2))^{-1}$, $x \in \mathbb{R}$.

(c) The Students distribution with parameter $\alpha \in \mathbb{N}$ and density function $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.

14. (Maximum domain of attraction of the Gumbel ditribution)
Check whether the following distributions belong to the maximum domain of attraction $MDA(\Lambda)$ of the Gumbel distribution.

    (a) The normal distribution $F(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$, $x \in \mathbb{R}$.

    (b) The exponential distribution with density function $f(x) = \lambda^{-1}\exp\{-\lambda x\}$, $x > 0$, for some parameter $\lambda > 0$.

    (c) The lognormal distribution with density function $f(x) = (2\pi x^2)^{-1/2}\exp\{-(\ln x)^2/2\}$, $x > 0$.

15. (a) Let X be a random variable with distribution function F. Derive an equation relating $VaR_\alpha(X)$, $CVaR_\alpha(X)$ and $e_X(q_\alpha)$, where $q_\alpha := VaR_\alpha(X)$ and $\alpha \in (0,1)$.

    (b) Compute the mean excess function of the exponential distribution, i.e. compute $e_X(u)$ for $X \sim Exp(\lambda)$ and $u \geq 0$.

    (c) Use the result form (a) to compute $CVaR_\alpha(X)$ in the case of the exponential distribution, i.e. for $X \sim Exp(\lambda)$, $\alpha \in (0,1)$.

16. Consider the daily relative returns (based on adjusted close prices) of the BMW and Siemens assets, *BMW.De* and *SIEGY*, respectively, over the time interval October 22, 2010, and October 19, 2020. Use the Hill estimator to get an approximation of the coefficient of the regular variation and determine the corresponding estimates for $\mathrm{VaR}_{0.90}$ and $\mathrm{CVaR}_{0.90}$ for each of these assets. Specify a plausible range of values of k to be chosen (depending on the sample size) and generate the Hill plot for those values of k. Based on the Hill plot make a suggestion for an appropriate value of k to be used and argue your choice carefully. Use `yahoo.finance.com` as a data source (see Exercise 8).

17. By means of the qq-plot check whether a normal distribution or a heavy tailed distribution like the (generalized) Pareto distribution is more appropriate to model the right tail of the loss distribution of the BMW and Siemens assets as described in Exercise 16, respectively. To this end you should compare the empirical quantiles of the above mentioned losses to the (numerically or analytically) computed quantiles of the reference distributions (i.e. a normal and a generalized Pareto distribution) and summarize the results graphically as described schematically in the lecture.

18. Consider the logarithmic daily returns of the close prises of the Nasdaq Composite Index (`^IXIC`) and apply the method of the Hill estimator to analyse their tails. Perform the following steps with three different time intervals: (I) from November 1, 1996 till November 2, 2018, (II) from November 1, 1996 till December 26, 2008, and (III) from December 29, 2008 till October 20, 2020. Compare the obtained results and comment upon your findings.

    (a) Compare the tails of the empirical distribution of the data set to the tails of the exponential ditribution by means of the QQ-plot.

    (b) Compute the Hill estimator for the empirical data. Argue carefully upon your choice of the threshold parameter k based on the inspection of the Hill plot as in the case of the fire insurance example discussed in the lecture.

    (c) Based on the Hill estimator give an estimator for the $\mathrm{VaR}_{0.95}$ and the $\mathrm{VaR}_{0.99}$ of the data set.

    The data can be downloaded from `finance.yahoo.com`: search for the required index (you can well search for the abbreviation given in paranthesis above), click 'Historical Data', update the 'Time Period' and 'Frequency' appropriately, and finally klick on 'Download data'.