Risk theory and management in actuarial science
Winter term 2020/21

fourth worksheet

22. (A coherent premium principle)

    Consider two constants $p > 1$ and $\alpha \in [0, 1)$. Let $(\Omega, \mathcal{F}, P)$ be some fixed probability space and $\mathcal{M}$ be the set of all random variables L on $(\Omega, \mathcal{F})$ for which $E(|L|^p)^{1/p}$ is finite, i.e. $E(|L|^p)^{1/p} < \infty$. Define a risk measure $\rho_{\alpha,p} := E(L) + \alpha||(L - E(L))^+||_p$ on $\mathcal{M}$, where $||X||_p := E(|X|^p)^{1/p}$ is the L^p-norm of the positive part of the centered random variable $X - E(X)$ for any random variable $X \in \mathcal{M}$. Show that $\rho_{\alpha,p}$ is a coherent risk measure for any $p > 1$ and any $\alpha \in [0, 1)$. So we get a whole family of coherent risk measures $\rho_{\alpha,p}$ for $p > 1$ and $\alpha \in [0, 1)$. How do the parameters α and p influence $\rho_{\alpha,p}$? Which parameter values lead to more "conservative" risk measures?

23. (Generalized scenarios as coherent risk measures)

    Denote by $\mathcal{P}$ a set of probability measures on some underlying measurable space $(\Omega, \mathcal{F})$ and set

    $$\mathcal{M}_{\mathcal{P}} := \{L\colon L \text{ is a r.v. on } (\Omega, \mathcal{F}),\ E^Q(|L|) < \infty \text{ for all } Q \in \mathcal{P}\},$$

    where $E^Q(X)$ denotes the expected value of a random variable X under the probability measure Q. Then *the risk measure induced by the set of generalized scenarios* $\mathcal{P}$ is the mapping $\rho_{\mathcal{P}}\colon \mathcal{M}_{\mathcal{P}} \to \mathbb{R}$ such that $\rho_{\mathcal{P}}(L) := \sup\{E^Q(L)\colon Q \in \mathcal{P}\}$. Show that $\rho_{\mathcal{P}}$ is coherent on $\mathcal{M}_{\mathcal{P}}$ for any set $\mathcal{P}$ of probability measures on $\mathcal{M}_{\mathcal{P}}$. Interprete the scenario based risk measures (cf. lecture) as a risk measure generalized by an appropriately defined set of probability measures on appropriately defined discrete probability spaces[1].

24. (a) Show that $W_d(u_1, u_2, \ldots, u_d == \max\{\sum_{i=1}^d u_i - d + 1, 0\}$ is indeed a lower bound for any copula $C\colon [0,1]^d \to [0,1]$, i.e. that $W_d(u_1, u_2, \ldots, u_d) \le C(u_1, u_2, \ldots, u_d)$ holds for any $d \in \mathbb{N}$, $d \ge 2$, any $(u_1, \ldots, u_d) \in [0,1]^d$ and any copula C as above.

    (b) Show that the Fréchet lower bound W_d is not a copula for $d \ge 3$.

    Hint: Show that the rectangle inequality

    $$\sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

    where $(a_1, a_2, \ldots, a_d)$, $(b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$ and $u_{k1} = a_k$ und $u_{k2} = b_k$ for all $k \in \{1, 2, \ldots, d\}$, is violated if $d \ge 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.

25. Let X_i, $i = 1, 2$, be two lognormally distributed random variables with $X_1 \sim \mathit{Lognormal}(0, 1)$ und $X_2 \sim \mathit{Lognormal}(0, \sigma^2)$, $\sigma > 0$. Compute $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ in dependence of σ and compare their values for different values of $\sigma > 0$. What can you say about the copula of (X_1, X_2) in each of the cases? Plot the graphs of $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ as functions of σ and comment on the behaviour of these functions for $\sigma \to +\infty$?

    Hint: Consider $X_1 = \exp(Z)$ and $X_2 = \exp(\sigma Z)$ or $X_2 = \exp(-\sigma Z)$ for a standard normally distributed random variable Z.

26. Construct two random vectors $(X_1, X_2)^T$ and $(Y_1, Y_2)^T$ with different joint distributions $F_{(X_1,X_2)}$, $F_{(Y_1,Y_2)}$, respectively, such that

    (a) the variables X_1, X_2, Y_1, Y_2 are standard normally distributed, i.e. $X_1, X_2, Y_1, Y_2 \sim N(0, 1)$,

[1] It can be shown that in the case of discrete probability spaces any coherent risk measure is induced by some set of generalized scenarios as described above, see Proposition 6.11 in *A.J. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management: Concepts, Techniques amd Tools , Princeton Unversity Press, 2005.*

(b) the two X-variables and the two Y-variables are uncorrelated, respectively, i.e. $\rho_L(X_1, X_2) = 0$, $\rho_L(Y_1, Y_2) = 0$, and

(c) the α-quantiles of the corresponding sums are different, i.e. $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds for some $\alpha \in (0,1)$, where $F_{X_1+X_2}$, $F_{Y_1+Y_2}$ are the distributions of X_1+X_2 and Y_1+Y_2, respectively.

Conclude that in general it is not possible to draw conclusions about the loss of a portfolio if only the loss distributions of the single assets in portfolio and their mutual linear correlation coefficients are known.

Hint: Choose (X_1, X_2) to be bivariate standard normally distributed, i.e. $(X_1, X_2) \sim N_2(0, I_2)$, where 0 denotes the zero vector in $\mathbb{R}^2$ and I_2 denotes the identity matrix in $\mathbb{R}^{2\times 2}$. Choose Y_1 to be standard normally distributed, $Y_1 \sim N(0,1)$, and set $Y_2 := VY_1$, where V is a discrete random variable independent on Y_1 with values 1 and -1 taken with probability 1/2 each.

27. (Co-monotonicity and anti-monotonicity)

(a) Let Z be a random variable with continuous cumulative distribution function F, $Z \sim F$. Let f_1, f_2 be to strictly monotone increasing functions on $\mathbb{R}$ and let f_3 be a strictly monotone decreasing function on $\mathbb{R}$. Let $X_i = f_i(Z)$, for $i = 1, 2, 3$. Show that the Fréchet upper bound M is a copula of (X_1, X_2) and the Fréchet lower bound W is a copula of (X_1, X_3).

(b) Let W be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$.

(c) Let M be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \stackrel{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$.