Risk theory and risk management in actuarial science
Winter term 2020/2021

5th work sheet

28. Prove the formula of Höffding

$$cov(X_1, X_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(F(x_1, x_2) - F_1(x_1)F_2(x_2))dx_1dx_2\,,$$

for any random vector $(X_1, X_2)^T$ with joint c.d.f. F and marginal c.d.f. F_1, F_2 for which $cov(X_1, X_2) < \infty$ holds.

Hint: Show first the identity $2cov(X_1, X_2) = E((X_1 - \bar{X}_1)(X_2 - \bar{X}_2))$, where $(\bar{X}_1, \bar{X}_2)$ is an i.i.d. copy of (X_1, X_2). Then use the identity $a - b = \int_{-\infty}^{+\infty}(\mathrm{I\!I}_{\{b\leq x\}} - \mathrm{I\!I}_{\{a\leq x\}})\mathrm{d}x$ and apply the latter to the pairs $X_1 - \bar{X}_1$ and $X_2 - \bar{X}_2$.

29. (a) Let $(X_1, X_2)^T$ be a t-distributed random vektor with ν degrees of freedom, expected value $(0, 0)$ and linear correlation coefficient matrix $\rho \in (-1, 1]$, i.e. $(X_1, X_2)^T \sim t_2(\vec{0}, \nu, R)$ where R is 2×2 matrix with 1 on the diagonal and ρ outside the diagonal. Show that the following equality holds for $\rho > -1$:

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$$

Hint: Use the fact (no need to prove it!) that conditional on $X_1 = x$ the following holds

$$\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2 - \rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}\,.$$

Recall the stochatic representation of the bivariate t-distribution as $\mu + \sqrt{W}AZ$, where Z is bivariate standard normally distributed and W is such that $\frac{\nu}{W} \sim \chi^2_\nu$ while being independent on Z (cf. lecture).

(b) Apply (a) to conclude that for a random vector with continuous marginal distributions $(X_1, X_2)^T$ and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and a correlation matrix R as in (a) the following equalities holds:

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\,.$$

30. Generate scatter plots of five thousand points each from the following four distributions (a) standard normal margins and copula $C^{Ga}_{\rho=0.5}$ (b) Student t margins with 4 degrees of freedom and copula $C^{Ga}_{\rho=0.5}$ (c) standard normal margins and copula $C^t_{\nu=4,\rho=0.5}$ (d) Student t margins with 4 degrees of freedom and copula $C^t_{\nu=4,\rho=0.5}$. Mark the 0.01, 0.0005, 0.09 and 0.995 quantiles by horizontal and vertical lines and comment on the pictures.

31. The Gumbel family C^{Gu}_θ and the Clayton family C^{Cl}_θ are two one-parametric families of copulas given as

$$C^{\mathrm{Gu}}_\theta(u_1, u_2) := \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right),\ \theta \geq 1, \text{ and}$$

$$C^{\mathrm{Cl}}_\theta(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta},\ \theta > 0.$$

(a) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependence for the copulas C^{Gu}_θ, C^{Cl}_θ, respectively.

(b) The independence copula Π is given by $\Pi(u_1,u_2) := u_1u_2$, for $(u_1,u_2)\in[0,1]^2$. Show that C_θ^{Gu} tends to the independence copula Π if θ tends to 1 and to the upper Fréchet bound M if θ tends to infinity. In this case we say that *the lower limit of the Gumbel copula is the independence copula* Π *and its upper limit is the Fréchet upper bound* M. Analogously show that the lower limit of the Clayton copula is the independence copula Π for $\theta\to 0^+$ and its upper limit is the Fréchet upper bound M for $\theta\to+\infty$. Now considerer an extension of the Clayton copula C_θ^{Cl} for $\theta\in[-1,0)$, defined as an Archimedian copula with generator $\phi_\theta(t)=\frac{1}{\theta}(t^{-\theta}-1)$ for $t\in(0,1]$ and $\phi_\theta(0)=+\infty$. Show that for $\theta=-1$ the Clayton copula C_{-1}^{Cl} coincides with the Fréchet lower bound W.

32. (a) Give an algorithm which generates a two-dimensional random vector with standard normal margin distributions, a given value $\rho_\tau\in[0,1)$ of Kendall's tau and (i) a Gumbel copula (ii) a Clayton copula.

(b) Apply the above algorithm to generate scatter plots of five thousand points each from the following two distributions (a) standard normal margins and copula C_θ^{Gu} and (b) standard normal margins and copula C_θ^{Cl}, with varying values of ρ_τ. The latter should be chosen such that the contrast between the Gumbel copula having just an upper tail dependence and the Clayton copula having just a lower tail dependence is clearly observable in the pictures.

33. **Archimedian Copulas**

(a) Show that for every $\theta\in\mathbb{R}\setminus\{0\}$ the function $\phi_\theta^{Fr}(t)=-\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ generates an Archmedian copula, the so-called Frank copula $C_\theta^{Fr}\colon[0,1]^2\to[0,1]$. Check that the following equality holds $\forall u_1,u_2\in[0,1]$:

$$C_\theta^{Fr}(u_1,u_2)=-\frac{1}{\theta}\ln\left(1+\frac{(\exp(-\theta u_1)-1)(\exp(-\theta u_2)-1)}{\exp(-\theta)-1}\right),\theta\in\mathbb{R}\setminus\{0\}\,.$$

(b) Show that for every $\theta>0$ and for every $\delta\geq 1$ the function $\phi_{\theta,\delta}^{GC}(t)=\theta^{-\delta}(t^{-\theta}-1)^\delta$ generates an Archmedian copula, the so-called *generalized Clayton copula* $C_{\theta,\delta}^{GC}\colon[0,1]^2\to[0,1]$. Check that the following equality holds $\forall u_1,u_2\in[0,1]$:

$$C_{\theta,\delta}^{GC}(u_1,u_2)=\{[(u_1^{-\theta}-1)^\delta+(u_2^{-\theta}-1)^\delta]^{1/\delta}+1\}^{-1/\theta},\ \theta\geq 0\,,\delta\geq 1\,.$$

(c) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependency for the copulas C_θ^{Fr} and $C_{\theta,\delta}^{GC}$, respectively.

34. **Asymmetric bivariate copulas**

Let C_θ be any exchangeable bivariate copula. Then a parametric family of asymmetric copulas $C_{\theta,\alpha,\beta}$ is obtained by setting

$$C_{\theta,\alpha,\beta}(u_1,u_2)=u_1^{1-\alpha}u_2^{1-\beta}C_\theta(u_1^\alpha,u_2^\beta)\,,\ 0\leq u_1,u_2\leq 1, \tag{1}$$

where $0\leq\alpha,\beta\leq 1$. When C_θ is an Archimedian copula we refer to the copulas constructed by (1) as asymmetric bivariate Archimedian copulas.

Check that $C_{\theta,\alpha,\beta}$ defined as above is a copula by constructing a random vector with distribution function $C_{\theta,\alpha,\beta}$ and observing that its margins are standard uniform on $[0,1]$. Show that such a random vector (U_1,U_2) can be generated as follows:

(i) Generate a random pair (V_1,V_2) with distribution function C_θ.

(ii) Generate, independently of V_1, V_2, two independent standard uniform variables $\bar{U}_1$ and $\bar{U}_2$.

(iii) Set $U_1=\max\left\{V_1^{1/\alpha},\bar{U}_1^{1/(1-\alpha)}\right\}$ and $U_2=\max\left\{V_2^{1/\beta},\bar{U}_2^{1/(1-\beta)}\right\}$

(a) Observe that $C_{\theta,\alpha,\beta}$ is not exchangeable in general and give a condition on the its parameters which if fulfilled leads to an exchangeable copula $C_{\theta,\alpha,\beta}$. Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha=\beta=0$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha=\beta=1$?

(b) Consider $C_4 := C_4^{Gu}$ and generate a scatter plot of 10000 points of the asymmetric bivariate copula $C_{4,0.95,0.7}$ by applying the algorithm described in $(a) - (c)$ above.

(c) Generate all pairwise scatterplots with 5000 points each for a three dimensional Clayton copula C_2^{Cl}.

35. **Three-dimensional non-exchangeable Archimedean copulas**
Suppose that ϕ_1 and ϕ_2 are two *strict* generators of Archimedean copulas, i.e. ϕ_1 and ϕ_2 are completely monotonic decreasing functions for which the pseudo-inverse function coincides with the ordinary inverse function. Consider

$$C(u_1, u_2, u_3) = \phi_2^{-1}(\phi_2 \circ \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3)). \quad (2)$$

If ϕ_1^{-1}, ϕ_2^{-1} and $\phi_2 \circ \phi_1^{-1}$ are completely monotonic decreasing functions mapping $[0, \infty]$ to $[0, \infty]$, then C defined by (2) is a copula.

Let (U_1, U_2, U_3) be a random vector with distribution function C as defined by (2).

(a) Show that if $\phi_1 \neq \phi_2$, then only U_1 and U_2 are exchangeable, i.e. $(U_1, U_2, U_3) \stackrel{d}{=} (U_2, U_1, U_3)$, but no other swapping of subscritps is possible. Show moreoever that if $\phi_1 = \phi_2$, than C is an exchangeable copula.

(b) Show that all bivariate margins of C defined by (2) are themselves Archimedean copulas; the margins C_{13} (obtained by components 1 and 3) and C_{23} (obtained by components 2 and 3) have generator ϕ_2, whereas the margin C_{12} (obtained by components 1 and 2) has generator ϕ_1.

36. Try to generate an appropriate model for the bivariate distribution of the daily log returns of BMW and Siemens assets for the time interval January 1, 2010 to December 1, 2020. Address in particluar the following issues: margin distributions, symmetry, upper and/or lower tail dependence, linear correlation and rank correlations coefficients. Include also the inspection of the graphical representation of the data in your reasoning.