Risk theory and risk management in actuarial science
Winter term 2020/2021

6th work sheet

37. **Equivalent threshold models**
Let $X = (X_1, X_2, \ldots, X_m)'$ be an m-dimensional random vector and let $D \in \mathbb{R}^{m\times n}$ be a deterministic matrix with elements d_{ij} such that for every i, $1 \le i \le m$, the elements of the i-th row form a set of increasing thresholds satisfying $d_{i,1} < d_{i2} \ldots < d_{in}$. Introduce additionally $d_{i0} = -\infty$, $d_{i,n+1} = +\infty$ and set
$$S_i = j \Longleftrightarrow d_{ij} < X_i \le d_{i,j+1}\,, \text{for } j \in \{0, \ldots, n\}, i \in \{1, \ldots, m\}.$$
Then (X, D) is said to define a *threshold model for the state vector* $S = (S_1, \ldots, S_m)'$. We refere to X as the *vector of critical variables* and denote its marginal distribution functions by $F_i(x) = P(X_i \le x)$, for $i \in \{1, 2, \ldots, m\}$. The i-th row of D contains the critical thresholds for firm i. By definition, default (corresponding to event $S_i = 0$) occurs iff $X_i \le d_{i1}$, thus the default probability of company i is given by $\bar{p}_i := F_i(d_{i1})$. Let Y_i be the default indicator of company i, i.e. $Y_i \in \{0, 1\}$ with $Y_i = 1$ iff company 1 defaults, hence $Prob(Y_i = 1) = \bar{p}_i$ and $Prob(Y_i = 0) = 1 - \bar{p}_i$, for $1 \le i \le m$. We denote by $\rho(Y_i, Y_j)$ the *default correlation* of two firms $i \ne j$; this quantity depends on $E(Y_i, Y_j)$ (how?) which in turn depends on the joint distribution of (X_i, X_j), and hence on the copula of $(X_1, X_2, \ldots, X_m)'$. (Notice that in general the latter is not fully determined by the *asset correlation* $\rho(X_i, X_j)$.)

Two threshold models (X, D) and $(\tilde{X}, \tilde{D})$ for the state vectors S and $\tilde{S}$, respectively, are called equivalent, iff S and $\tilde{S}$ have the same probability distribution.

Show that two threshod models (X, D) and $(\tilde{X}, \tilde{D})$ with state vectors S and $\tilde{S}$, respectively, are equivalent if the following conditions hold:

(a) The marginal distributions of the random vectors S and $\tilde{S}$ coincide, i.e. $P(S_i = j) = P(\tilde{S}_i = j)$, for all $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$.
(b) X and $\tilde{X}$ admit the same copula C.

38. A bank has a loan portfolio of 100 loans. Let X_k be the default indicator for loan k such that $X_k = 1$ in case of default and 0 otherwise, for $k \in \{1, \ldots, 100\}$.

(a) Supoose that X_k are independent and identically distributed with $P(X_k = 1) = 0.01$. Compute the expected value $E(N)$ of the number N of defaults and $P(N = k)$ for $k \in \{0, 1, \ldots, 100\}$.
(b) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z$, where $P(Z = 0.01) = 0.9$ and $P(Z = 0.11) = 0.1$. Compute the expected value $E(N)$ where N is defined as in (a).
(c) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z^9$, where Z is uniformly distributed on $(0, 1)$. Compute the expected value $E(N)$ where N is defined as in (a).

39. An m-dimensional random vector X is said to have a p-dimensional conditional independence structure with conditioning variable Ψ iff there is some $p \in \mathbb{N}$, $p < m$, and a p-dimensional random vector $\Psi = (\Psi_1, \ldots, \Psi_p)^t$, such that the random variables X_1,...,X_m are independent conditional on Ψ. Consider a threshold model (X, D) as defined in Exercise 37 and assume that X has a p-dimensional conditional independence structure with conditioning variable Ψ. Show that the default indicators $Y_i = \mathbb{I}_{X_i \le d_{i1}}$ follow a Bernoulli mixture model with factor Ψ. How are given the conditional default probabilities for this Bernoulli mixture model?

40. Suppose that the critical variables $X = (X_1, \ldots, X_m)^t$ have a normal mean-variance mixture distribution, i.e. $X = m(W) + \sqrt{W}Z$ with an m-dimensional random vector Z, a positive, scalar random variable W independent of Z, and a measurable function $m\colon [0; +\infty) \to \mathrm{I\!R}^{\mathrm{m}}$. Assume that Z (and hence X) follows a linear factor model of the form $Z = BF + \epsilon$, where $F \sim N_p(\vec{0}, \Omega)$ is a p-dimensional normally distributed vector with expected value $\vec{0} \in \mathrm{I\!R}^{\mathrm{p}}$ and covariance matrix $\Omega \in \mathrm{I\!R}^{\mathrm{p\times p}}$, $B \in \mathrm{I\!R}^{\mathrm{m\times p}}$ is a deterministic loading matrix, and the components $\epsilon_1,\ldots,\epsilon_m$ of ϵ are i.i.d. normally distributed random variables which are also independent of F. Show that F hast a $(p+1)$-conditional independence structure (see the definition in Exercise 39). How are given the conditional default probabilities for the corresponding Bernoulli mixture model of the default indicators Y_i in this case (cf. Exercise 39)?

41. (Application of Archimedian copulas in threshold models)

    Consider a threshold model (X, D) where X has an Archimedian copula C with generator ϕ such that ϕ^{-1} is the Laplace transform of some nonnegative distribution function G with $G(0) = 0$. Let $d = (d_{11}, \ldots, d_{m1})^t$ denote the first column of D containing the default thresholds. We write then (X, d) for a threshold model of default with an Archimedian copula dependence and denote by $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_m)^t$ the vector of default probabilities, where $\bar{p}_i = \mathrm{I\!P}(\mathrm{X_i} \leq \mathrm{d_{i1}})$, for $i \in \{1, 2, \ldots, m\}$. Consider a nonnegative random variable $\Psi \sim G$ and random variables $U_1,\ldots,U_m$ that are conditionally independent given Ψ with conditional distribution function $\mathrm{I\!P}(\mathrm{U_i} \leq \mathrm{u}|\Psi = \psi) = \exp(-\psi\phi(\mathrm{u}))$, for $u \in [0, 1]$. Check that $U = (U_1, \ldots, U_m)^t$ has distribution function C. Show that (X, d) and $(U, \bar{p})$ are two equivalent threshold models for default (cf. Exercise 37). How are given the conditional default probabilities $p_i(\Psi)$ in this case?

    Consider the Clayton copula C_θ^{Cl} with generator function $\phi(t) = t^{-\theta} - 1$ and assume that we want to construct a Bernoulli mixture model that is equivalent to a threshold model driven by C_θ^{Cl}. In this Bernoulli mixture model all conditional default probabilites $p_i(\Psi)$ would coincide; such a Bernoulli mixture model is called *exchangeable*. Assume moreover that the probability of default for any creditor is given by π and the probability that an arbitrary pair of creditors defaults is given by π'. What value of θ would lead to the required exchangeable Bernoulli mixture modell?

42. Apply the CreditRisk$^+$ approach for a credit portfolio with $n = 100$ credits, and m risk factors, where $m = 1$ or $m = 5$. Consider the settings $\bar{\lambda}_i = \bar{\lambda} = 0.15$, $\alpha_j = \alpha = 1$, $\beta_j = \beta = 1$, $a_{i,j} = 1/m$, for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Let N be the number of defaulting creditors. Recall that $\mathrm{I\!P}(\mathrm{N} = \mathrm{k}) = \frac{1}{\mathrm{k}!}\mathrm{g_N^{(k)}}(0) = \frac{1}{\mathrm{k}!}\frac{\mathrm{d^k g_N(0)}}{\mathrm{dt^k}}$ holds for any $k \in \overline{1..n}$, where g_N is the probability generating function (pgf) of N (cf. the lecture).

    (a) Based on the closed form expressions discussed in the lecture derive concrete formulas for the probability generating functions $\tilde{g}_N(t)$ and $\bar{g}_N(t)$ of N in the cases $m = 1$ and $m = 5$, respectively.

    (b) Show that the following recursive formula holds for $g_N = \tilde{g}_N$ and $g = \bar{g}_N$ and $k > 1$:

    $$g_N^{(k)}(0) = \sum_{l=0}^{k-1} \binom{k-1}{l} g_N^{(k-1-l)}(0) \sum_{j=1}^{m} l!\alpha_j\delta_j^{l+1}\,.$$

    (c) Compute and plot $\mathrm{I\!P}(\mathrm{N} = \mathrm{k})$, for $k \in \overline{1..n}$, in both cases $m = 1$ and $m = 5$. Compare the two plots and interprete the results.