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Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlaton matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.

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In the bivariate case we have:

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{\frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)}\right\} dx_1 dx_2,$$

where $\rho \in (-1, 1)$.

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$$C^t_{\alpha,R}(u) = t^d_{\alpha,R}(t^{-1}_\alpha(u_1), \ldots, t^{-1}_\alpha(u_d)).$$

$R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$, $i, j = 1, 2 \ldots, d$, is the correlation matrix of Z,

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In the bivariate case ($d = 2$):

$$C^t_{\alpha,R}(u_1, u_2) = \int_{-\infty}^{t_\alpha^{-1}(u_1)} \int_{-\infty}^{t_\alpha^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\alpha(1-\rho^2)}\right\}^{-\frac{\alpha+2}{2}} dx_1 dx_2.$$

for $\rho \in (-1, 1)$. R_{12} is the linear correlation coefficient of the corresponding bivariate t_α-distribution for $\alpha > 2$.

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The Gumbel and Clayton Copulas are not radial symmetric. Why?

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Let C be the copula of a distribution F with marginal distributions F_1,...,F_d. By differentiating

$$C(u_1, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$$

we obtain the density c of C:

$$c(u_1, \ldots, u_d) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \ldots f_d(F_d^{-1}(u_d))}$$

where f is the density function of F, f_i are the marginal density functions, and F_i^{-1} are the inverse functions of F_i, for $1 \leq i \leq d$,

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Examples of exchangeable copulas:

Gumbel, Clayton, and also the Gaussian copula C_P^{Ga} and the t-Copula $C_{\nu,P}^t$, if P is an *equicorrelation matrix*, i.e. $R = \rho J_d + (1 - \rho) I_d$.
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For bivariate exchangeable copulas we have:

$$P(U_2 \leq u_2 | U_1 = u_1) = P(U_1 \leq u_2 | U_2 = u_1).$$

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Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

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Theorem: Let $(X_1, X_2)^T \sim t_2(0, \nu, R)$ be a random vector with a t-distribution and ν degrees of freedom, expectation 0 and linear correlation matrix R. For $R_{12} > -1$ we have

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The proof is similar to the proof of the analogous theorem about the Gaussian copulas.
Hint:

$$X_2|X_1 = x \sim \left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}$$

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See McNeil et al. (2005) for a proof of the three last results.

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$$\phi(\phi^{[-1]}(t) = \begin{cases} t & 0 \leq t \leq \phi(0) \\ \phi(0) & \phi(0) \leq t \leq +\infty \end{cases}$$

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