Risk theory and risk management in actuarial science
Winter term 2017/18

8th work sheet

41. An m-dimensional random vector X is said to have *a p-dimensional conditional independent structure with conditioning variable* Ψ iff there is some $p \in \mathbb{N}$, $p < m$ and a p-dimensional random vector $\Psi = (\Psi_1, \ldots, \Psi_p)^t$, such that the random variables X_1,...,X_m are independent conditional on Ψ. Consider a threshold model (X, D) as defined in Exercise 37 and assume that X has a p-dimensional conditional independent structure with conditioning variable Ψ. Show that the default indicators $Y_i = \mathbb{I}_{\{X_i \leq d_{i1}\}}$ follow a Bernoulli mixture model with factor Ψ. How are given the conditional default probabilities for this Bernoulli mixture model?

42. Suppose that the critical variables $X = (X_1, \ldots, X_m)^t$ have a *normal mean-variance mixture distribution*, i.e. $X = m(W) + \sqrt(W)Z$ with an m-dimensional random vector Z, a positive, scalar random variable W independent of Z, and a measurable function $m\colon [0, +\infty) \to \mathbb{R}^m$. Assume that Z (and hence X) follows a linear factor model of the form $Z = BF + \varepsilon$, where $F \sim N_p(\vec{0}, \Omega)$ is p-dimensional normally distributed vector with expected value $\vec{0} \in \mathbb{R}^p$ and covariance matrix $\Omega \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{m \times p}$ is a deterministic loading matrix, and the components ε_1,...,ε_m of ε are i.i.d. normally distributed random variables which are also independent of F. Show that F hast a $(p+1)$-conditional independence structure (see the definition in Exercise 41). How are given the conditional default probabilities for the corresponding Bernoulli mixture model of the default indicators Y_i in this case (cf. Exercise 41)?

43. (Application of Archimediam copulas in threshold models)
Consider a threshold model (X, D) where X has an Archimedian copula C with generator ϕ such that ϕ^{-1} is the Laplace transform of some nonnegative distribution function G with $G(0) = 0$. Let $d = (d_{i1}, \ldots, d_{mi})^t$ denote the first column of D containing the default thresholds. We write then (X, d) for a threshold model of default with an Archimedian copula dependence and denote by $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_m)^t$ the vector of default probabilities, where $\bar{p}_i = \mathbb{P}(X_i \leq d_{i1})$ for $i \in \{1, 2, \ldots, m\}$. Consider a nonnegative random variable $\Psi \sim G$ and random variables U_1,...,U_m that are conditionally independent given Ψ with conditional distribution function $\mathbb{P}(U_i \leq u | \Psi = \psi) = \exp(-\psi\phi(u))$ for $u \in [0, 1]$. Check that $U = (U_1, \ldots, U_m)^t$ has distribution function C. Show that (X, d) and $(U, \bar{p})$ are two equivalent threshold models for default (cf. exercise 37). How are given the conditional default probabilities $p_i(\Psi)$ in this case?

    Consider the Clayton copula C_θ^{Cl} with generator function $\phi(t) = t^{-\theta} - 1$ and assume that we want to construct a Bernoulli mixture model that is equivalent to a threshold model driven by C_θ^{Cl}. In this Bernoulli mixture model all conditional default probabilites $p_i(\Psi)$ would coincide; such a Bernoulli mixture model is called *exchangeable*. Assume moreover that the probability of default for any creditor is given by π and the probability that an arbitrary pair of creditors default is given by π_2. What value of θ would lead to the required exchangeable Bernoulli mixture modell?