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$e_1 = \frac{\ln(V_{A,i}(t) - \ln K_i + (r + \sigma_{A,i}^2/2)(T-t)}{\sigma_{A,i}(T-t)}$, $e_2 = e_1 - \sigma_{A,i}(T-t)$,

ϕ is the the standard normal distribution function and r is the risk free interest rate.

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$DD_i = \frac{\ln V_{A,i}(t) - \ln K_i + (\mu_{A,i} - \frac{\sigma_{A,i}^2}{2})(T-t)}{\sigma_{A,i}\sqrt{T-t}}$.

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$DD_i = \frac{\ln V_{A,i}(t) - \ln K_i + (\mu_{A,i} - \frac{\sigma^2_{A,i}}{2})(T-t)}{\sigma_{A,i}\sqrt{T-t}}$.

Then $P(V_{A,i}(T) < K_i) = P(Y_i < -DD_i)$ and in the general setup of the latent variable model with $m = 1$ we have $d_{i1} = -DD_i$.

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Summary of the univariate KMV model to compute the default probability of a company:

- Estimate the asset value $V_{A,i}$ and the volatilty $\sigma_{A,i}$ by using observations of the market value and the volatility of equity $V_{E,i}$, $\sigma_{E,i}$, the book of liabilities K_i, and by solving the system of equations above.
- Compute the distance-to-default DD_i by means of the corresponding formula.
- Estimate the default probability p_i in terms of the empirical distribution which relates the distance to default with the expected default frequency.

The multivariate KMV model: computation of multivariate default probabilities

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$\mu_{A,i}$ is the drift, $\sigma^2_{A,i} = \sum_{j=1}^{m} \sigma^2_{A,i,j}$ is the volatility, and $\sigma_{A,i,j}$ quantifies the impact of the jth Brownian motion on the asset value of firm i.

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Set $Y_i := \frac{\sum_{j=1}^m \sigma_{A,i,j}(W_j(T)-W_j(t))}{\sigma_{A,i}\sqrt{T-t}}$. Then $Y = (Y_1, Y_2, \ldots, Y_n) \sim N(0, \Sigma)$,

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We get $V_{A,i}(T) < K_i \iff Y_i < -DD_i$ with

$$DD_i = \frac{\ln V_{A,i}(t) - \ln K_i + \left(\frac{-\sigma_{A,i}^2}{2} + \mu_{A,i}\right)(T-t)}{\sigma_{A,i}\sqrt{T-t}}.$$

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The probability that the k first firms default:

$$P(X_1 = 1, X_2 = 1, \ldots, X_k = 1) = P(Y_1 < -DD_1, \ldots, Y_k < -DD_k)$$
$$= C_\Sigma^{Ga}(\phi(-DD_1), \ldots, \phi(-DD_k), 1, \ldots, 1),$$

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Joint default frequency:

$JDF_{1,2,\ldots,k} = C_\Sigma^{Ga}(EDF_1, EDF_2, \ldots, EDF_k, 1, \ldots, 1),$
where EDF_i is the default frequency for firm i, $i = 1, 2, \ldots, k$.

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$Y = (Y_1, Y_2, \ldots, Y_n)^T = AZ + BU$ where
$Z = (Z_1, \ldots, Z_k)^T \sim N_k(0, \Lambda)$ are the k common factors,
$U = (U_1, \ldots, U_n)^T \sim N_n(0, I)$ are the company specific factors such that Z and U are independent, and the constant matrices $A = (a_{ij}) \in \mathbb{R}^{n\times k}$, $B = diag(b_1, \ldots, b_n) \in \mathbb{R}^{n\times n}$ are model parameters.

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Then we have $cov(Y) = A\Lambda A^T + D$ where $D = diag(b_1^2, \ldots, b_n^2) \in \mathbb{R}^{n\times n}$.

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Let P be a portfolio consisting of n credits with a fixed holding duration (eg. 1 year). Let S_i be the status variable for debtor i, where the states are $0, 1, \ldots, m$ and $S_i = 0$ corresponds to default.

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Example: Rating system of Standard and Poor's
$m = 7$; $S_i = 0$ means default; $S_i = 1$ or CCC; $S_i = 2$ or B; $S_i = 3$ or BB; $S_i = 4$ or BBB; $S_i = 5$ or A; $S_i = 6$ or AA; $S_i = 7$ or AAA.

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Original state category	state category at the end of the year AAA	AA	A	BBB	BB	B	CCC	default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

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Recovery rates

In case of default the recovery rate depends on the status category of the defaulting debtor (prior to default). The mean and the standard deviation of the recovery rate are computed based on the historical data observed over time within each state category.

Evaluation of bonds if the status category changes

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Example: Consider a BBB bond with maturity 5 years, a nominal value of 100 units and a coupon of 6% each year.
The forward *forward yield curves* for each status category are given as follows (in %):

Status	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.73	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
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The bond pays 6 units at the end of the 4 years 1, 2, 3, 4 and 106 unit at the end of year 5.

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The bond pays 6 units at the end of the 4 years 1, 2, 3, 4 and 106 unit at the end of year 5.

Assumption: At the end of the first year the bond is rated as an A bond. The value at the end of the first year:

$$V = 6 + \frac{6}{1+3,73\%} + \frac{6}{(1+4,32\%)^2} + \frac{6}{(1+4,93\%)^3} + \frac{106}{(1+5,32\%)^4} = 108.64$$

Evaluation of bonds if the status category changes (contd.)

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Status category at the end of the first year	value
AAA	109.35
AA	109.17
A	108.64
BBB	107.53
BB	102.01
B	98.09
CCC	83.63
Default	51.13

Use the transition probabilities of the Markov chain (estimated in terms of historical data) to compute the expected value of the bond at the end of the first year.

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Joint probabilities of status category changes, e.g.

$$P(S_1 = 0, \ldots, S_n = 3) = P(Y_1 \leq d_{Def}, \ldots, d_B < Y_n \leq d_{BB})$$

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Use simulation to compute the risk measures (VaR, CVaR) of the bond portfolio, e.g. by generating a large number of scenarios and then computing the empirical estimators of VaR, CVaR.

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The 0-1 random vector $X = (X_1, \ldots, X_n)^T$ has a *Bernoulli mixture distribution (BMD)* iff there exists a random vector $Z = (Z_1, Z_2, \ldots, Z_m)^T$, $m < n$, and the functions $f_i \colon \mathbb{R}^m \to [0,1]$, $i = 1, 2, \ldots, n$, such that X conditioned on Z has independent components with $X_i | Z \sim \text{Bernoulli}(f_i(Z))$.

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If all function f_i coincide, i.e. $f_i = f$, $\forall i$, we get $N|Z \sim \text{Bin}(n, f(Z))$ for the number $N = \sum_{i=1}^n X_i$ of defaults.

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If $\lambda_i(Z) << 1$ we get for the number $\tilde{N} = \sum_{i=1}^n \bar{X}_i \approx \sum_{i=1}^n X_i$ of defaults:

$$\tilde{N}|Z \sim Poisson(\bar{\lambda}(Z)), \text{ where } \bar{\lambda} = \sum_{i=1}^n \lambda_i(Z).$$

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The density function of Z_j is given as $f_j(z) = \frac{z^{\alpha_j - 1} \exp\{-z/\beta_j\}}{\beta_j^{\alpha_j} \Gamma(\alpha_j)}$

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The density function of Z_j is given as $f_j(z) = \frac{z^{\alpha_j - 1} \exp\{-z/\beta_j\}}{\beta_j^{\alpha_j} \Gamma(\alpha_j)}$

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CreditRisk$^+$ - a Poisson mixture model

(developed by CSFB in 1997, see Crouhy et al. (2000) and
www.credit_suisse.com/investment_banking/research/en/credit_risk.jsp

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The goal: approximate the loss disribution by a discrete distribution and derive the generator function for the latter.

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(v) Let $g_X(t)$ be the pgf of X. Then $P(X=k) = \frac{1}{k!} g_X^{(k)}(0)$, where $g_X^{(k)}(t) = \frac{d^k g_X(t)}{dt^k}$.

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The loss function is then given by $L = \sum_{i=1}^n \bar{X}_i v_i L_0 \approx \sum_{i=1}^n X_i v_i L_0$, where $\bar{X}_i$ is the loss indicator and $(X_1, \ldots, X_n)$ has a PMD with factor vector $(Z_1, Z_2, \ldots, Z_m)$ as described above.

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$X_i|Z \sim Poi(\lambda_i(Z))$, $\forall i \Longrightarrow g_{X_i|Z}(t) = \exp\{\lambda_i(Z)(t-1)\}$, $\forall i \Longrightarrow$
$g_{N|Z}(t) = \prod_{i=1}^n g_{X_i|Z}(t) = \prod_{i=1}^n \exp\{\lambda_i(Z)(t-1)\} = \exp\{\mu(t-1)\}$,
with $\mu := \sum_{i=1}^n \lambda_i(Z) = \sum_{i=1}^n \left(\bar{\lambda}_i \sum_{j=1}^m a_{ij} Z_j\right)$.

The pgf of the loss distribution (contd.)

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Then

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Analogous computations as in the case of $g_N(t)$ yield:

$$g_L(t) = \prod_{j=1}^{m} \left(\frac{1-\delta_j}{1-\delta_j \Lambda_j(t)} \right)^{\alpha_j} \text{ wobei } \Lambda_j(t) = \frac{1}{\mu_j} \sum_{i=1}^{n} \bar{\lambda}_i a_{ij} t^{v_i}.$$

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Assume that $\bar{\lambda}_i = \bar{\lambda} = 0.15$, for $i = 1, 2, \ldots, n$, $\alpha_j = \alpha = 1$, $\beta_j = \beta = 1$, $a_{i,j} = 1/m$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

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For the computation of $P(N = k)$, $k = 0, 1, \ldots, 100$, we can use the following recursive formula

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Assume that $\bar{\lambda}_i = \bar{\lambda} = 0.15$, for $i = 1, 2, \ldots, n$, $\alpha_j = \alpha = 1$, $\beta_j = \beta = 1$, $a_{i,j} = 1/m$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

The probability that k creditors will default is given as follows for any $k \in \mathbb{N} \cup \{0\}$:
$P(N = k) = \frac{1}{k!} g_N^{(k)}(0) = \frac{1}{k!} \frac{d^k g_N}{dt^k}$.

For the computation of $P(N = k)$, $k = 0, 1, \ldots, 100$, we can use the following recursive formula

$g_N^{(k)}(0) = \sum_{l=0}^{k-1} \binom{k-1}{l} g_N^{(k-1-l)}(0) \sum_{j=1}^{m} l! \alpha_j \delta_j^{l+1}$, where $k > 1$.