Conditional Value at Risk (contd.)
Example 1:

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

(b) Let the distribution function F_L of the loss function L be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0, 1)$. Compute $CVaR_\alpha(L)$.

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Example 2:

Let $L \sim N(0, 1)$. Let ϕ und Φ be the density and the distribution function of L, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

Let $L' \sim N(\mu, \sigma^2)$. Show that $CVaR_\alpha(L') = \mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

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Exercise:
Let the loss L be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of L is

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where t_ν is the distribution function of L.

Methods for the computation of VaR und CVaR

Consider the portfolio value $V_m = f(t_m, Z_m)$, where Z_m is the vector of risk factors.

Let the loss function over the interval $[t_m, t_{m+1}]$ be given as $L_{m+1} = l_{[m]}(X_{m+1})$, where X_{m+1} is the vector of the risk factor changes, i.e.

$$X_{m+1} = Z_{m+1} - Z_m.$$

Consider observations (historical data) of risk factor values $Z_{m-n+1}, \ldots, Z_m$.
How to use these data to compute/estimate $VaR(L_{m+1})$, $CVaR(L_{m+1})$?

The empirical VaR and the empirical CVaR

Let $x_1, x_2, \dots, x_n$ be a sample of i.i.d. random variables $X_1, X_2, \dots, X_n$ with distribution function F.

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The empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{[x_k, +\infty)}(x)$$

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Assumption: $x_1 > x_2 > \ldots > x_n$. Then $q_\alpha(F_n) = x_{[n(1-\alpha)]+1}$ holds, where $[y] := \sup\{n \in \mathbb{N} \colon n \leq y\}$ for every $y \in \mathbb{R}$.

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Lemma

Let $\hat{q}_\alpha(F) := q_\alpha(F_n)$ and let F be a strictly increasing function. Then $\lim_{n\to\infty} \hat{q}_\alpha(F) = q_\alpha(F)$ holds $\forall \alpha \in (0,1)$, i.e. the estimator $\hat{q}_\alpha(F)$ is consistent.

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The empirical estimator of CVaR is $\widehat{\mathrm{CVaR}}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_k}{[(n(1-\alpha)]+1}$

A non-parametric bootstrapping approach to compute the confidence interval of the estimator

Let $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution function F and let $x_1 > x_2 > \ldots > x_n$ be an ordered sample of F.

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Goal: computation of an estimator of a certain parameter θ depending on F, e.g. $\theta = q_\alpha(F)$, and the corresponding confidence interval.

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Let $\hat{\theta}(x_1, \ldots, x_n)$ be an estimator of θ, e.g. $\hat{\theta}(x_1, \ldots, x_n) = x_{[(n(1-\alpha)]+1}$
$\theta = q_\alpha(F)$.
The required confidence interval is an (a, b) with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level p.

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The required confidence interval is an (a, b) with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level p.

Case I: F is known.
Generate N samples $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}$, $1 \le i \le N$, by simulation from F (N should be large)
Let $\tilde{\theta}_i = \hat{\theta}\left(\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}\right)$, $1 \le i \le N$.

Case I (cont.)

The empirical distribution function of $\hat{\theta}(x_1, x_2, \dots, x_n)$ is given as

$$F_N^{\hat{\theta}} := \frac{1}{N} \sum_{i=1}^{N} I_{[\tilde{\theta}_i, \infty)}$$

and it tends to $F^{\hat{\theta}}$ for $N \to \infty$.

The required conficence interval is given as

$$\left(q_{\frac{1-p}{2}}(F_N^{\hat{\theta}}), q_{\frac{1+p}{2}}(F_N^{\hat{\theta}}) \right)$$

(assuming that the sample sizes N und n are large enough).

Case II: F is not known ⇒ Bootstrapping!

The empirical distribution function of X_i, $1 \leq i \leq n$, is given as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i,\infty)}(x).$$

For n large $F_n \approx F$ holds.

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Generate samples from F_n be choosing n elementes in $\{x_1, x_2, \ldots, x_n\}$ and putting every element back to the set immediately after its choice

Assume N such samples are generated: $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \leq i \leq N$.

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.

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The empirical distribution of θ_i^* is given as $F_N^{\theta^*}(x) = \frac{1}{N}\sum_{i=1}^{N} I_{[\theta_i^*,\infty)}(x)$; it approximates the distribution function $F^{\hat{\theta}}$ of $\hat{\theta}(X_1, X_2, \ldots, X_n)$ for $N \to \infty$.

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A confidence interval (a, b) with confidence level p is given by

$$a = q_{(1-p)/2}(F_N^{\theta^*}),\; b = q_{(1+p)/2}(F_N^{\theta^*}).$$

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Thus $a = \theta^*_{[N(1+p)/2]+1}$, $b = \theta^*_{[N(1-p)/2]+1}$, where $\theta_1^* \ge \ldots \ge \theta_N^*$ is the sorted θ^* sample.

Summary of the non-parametric bootstrapping approach to compute confidence intervals

Input: Sample $x_1, x_2, \ldots, x_n$ of the i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function F and an estimator $\hat{\theta}(x_1, x_2, \ldots, x_n)$ of an unknown parameter $\theta(F)$, A confidence level $p \in (0, 1)$.

Output: A confidence interval I_p for θ with confidence level p.

- Generate N new Samples $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \leq i \leq N$, by chosing elements in $\{x_1, x_2, \ldots, x_n\}$ and putting them back right after the choice.
- Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.
- Setz $I_p := \left(\theta^*_{[N(1+p)/2]+1,N}, \theta^*_{[N(1-p)/2]+1,N}\right)$, where $\theta^*_{1,N} \geq \theta^*_{2,N} \geq \ldots \theta^*_{N,N}$ is obtained by sorting $\theta_1^*, \theta_2^*, \ldots, \theta_N^*$.

An approximative solution without bootstrapping

Input: A sample $x_1, x_2, \ldots, x_n$ of the random variables X_i, $1 \leq i \leq n$, i.i.d. with unknown continuous distribution function F, a confidence level $p \in (0, 1)$.

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Output: A $p' \in (0, 1)$, with $p \leq p' \leq p + \epsilon$, for some small ϵ, and a confidence interval (a, b) for $q_\alpha(F)$, i.e. $a = a(x_1, x_2, \ldots, x_n)$, $b = b(x_1, x_2, \ldots, x_n)$, such that

$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F) \leq (1-p)/2$ holds.

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Assume w.l.o.g. that the sample is sorted $x_1 \ge x_2 \ge \ldots \ge x_n$.

Determine $i > j$, $i, j \in \{1, 2, \ldots, n\}$, and the smallest $p' > p$, such that

$$P\left(x_{i,n} < q_\alpha(F) < x_{j,n}\right) = p' \qquad (*) \qquad \text{and}$$

$$P\left(x_i \ge q_\alpha(F)\right) \le (1-p)/2 \text{ and } P\left(x_j \le q_\alpha(F)\right) \le (1-p)/2 (**).$$

An approximative solution without bootstrapping (contd.)

Let $Y_\alpha := \#\{x_k : x_k > q_\alpha(F)\}$

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We get $P(x_j \leq q_\alpha(F)) \approx P(x_j < q_\alpha(F)) = P(Y_\alpha \leq j-1)$
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$Y_\alpha \sim Bin(n, 1-\alpha)$ since $Prob(x_k \geq q_\alpha(F)) \approx 1-\alpha$ for a sample point x_k.

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Compute $P(x_j \leq q_\alpha(F))$ and $P(x_i \geq q_\alpha(F))$ for different i and j until indices $i, j \in \{1, 2, \ldots, n\}$, $i > j$, which fulfill $(**)$ are found.

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Set $a := x_j$ and $b := x_i$.

Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$; the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

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Assumption: the historically realized losses are i.i.d.

The historically realized losses can be seen as a sample of the loss distribution. Sort the historical losses l_i, $1 \le i \le n$, to obtain $l_{1,n} \ge l_{2,n} \ge \ldots \ge l_{n,n}$.

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Empirical CVaR: $\widehat{CVaR} = \frac{\sum_{i=1}^{[n(1-\alpha)]+1} l_{i,n}}{[n(1-\alpha)]+1}$.

Analogously, we can consider the loss aggregated over a given time interval (number of days or general time units).

VaR and CVaR of the loss aggregated over a number of days, e.g. 10 days, over the days $m-n+10(k-1)+1, m-n+10(k-1)+2, \ldots, m-n+10(k-1)+10$, denoted by $l_k^{(10)}$ is given as

$$l_k^{(10)} = l_{[m]}\left(\sum_{j=1}^{10} x_{m-n+10(k-1)+j}\right) \qquad k = 1, \ldots, [n/10]$$

Historical simulation (contd.)

Advantages:

- simple implementation
- considers intrinsically the dependencies between the elements of the vector of the risk factors changes $X_{m-k} = (X_{m-k,1}, \ldots, X_{m-k,d})$.

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- simple implementation
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Disadvantages:

- lots of historical data needed to get good estimators
- the estimated loss cannot be larger than the maximal loss experienced in the past

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$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
$X_{m+1} = (X_{m+1,1}, X_{m+1,2}, \ldots, X_{m+1,d})^T$.

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$L^{\Delta}_{m+1} = l^{\Delta}_{m}(X_{m+1}) = -V\sum_{i=1}^{d} w_i X_{m+1,i} = -Vw^T X_{m+1}$,
where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
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Estimator for VaR: $\widehat{VaR}(L_{m+1}) = -Vw^T\hat{\mu} + V\sqrt{w^T\hat{\Sigma}w}\,\phi^{-1}(\alpha)$

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Advantages:

- analytical solution
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Disadvantages:

- Linearisation is not always appropriate, only for a short time horizon reasonable
- The normal distribution assumption could lead to underestimation of risks and should be argued upon (e.g. in terms of historical data)

Monte-Carlo approach

(1) historical observations of risk factor changes $X_{m-n+1}, \ldots, X_m$.

(2) assumption on a parametric model for the cumulative distribution function of X_k, $m-n+1 \leq k \leq m$;
e.g. a common distribution function F and independence

(3) estimation of the parameters of F.

(4) generation of N samples $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ from F $(N \gg 1)$ and computation of the losses $l_k = l_{[m]}(\tilde{x}_k)$, $1 \leq k \leq N$

(5) computation of the empirical distribution of the loss function L_{m+1}:

$$\hat{F}_N^{L_{m+1}}(x) = \frac{1}{N}\sum_{k=1}^{N} I_{[l_k,\infty)}(x).$$

(5) computation of estimates for the VaR and CVAR of the loss function: $\widehat{VaR}(L_{m+1}) = \left(\hat{F}_N^{L_{m+1}}\right) = l_{[N(1-\alpha)]+1,N}$,
$\widehat{CVaR}(L_{m+1}) = \frac{\sum_{k=1}^{[N(1-\alpha)]+1} l_{k,N}}{[N(1-\alpha)]+1}$,
where the losses are sorted $l_{1,N} \geq l_{2,N} \geq \ldots \geq l_{N.N}$.

Monte-Carlo approach (contd.)

Advantages:

- very flexible; can use any distribution F from which simulation is possible
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Disadvantages:

- computationally expensive; a large number of simulations needed to obtain good estimates

Monte-Carlo approach (contd.)

Example

The portfolio consists of one unit of asset S with price be S_t at time t. The risk factor changes

$$X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k}),$$

are i.i.d. with distribution function F_θ for some unknown parameter θ.

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Depending on F_θ it can be complicated or impossible to compute CVaR analytically.
Alternative: Monte-Carlo simulation.

Monte-Carlo approach (contd.)

Example

Let the portfolio and the risk factor changes X_{k+1} be as in the previous example.
A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

$$\begin{aligned} X_{k+1} &= \sigma_{k+1} Z_{k+1} && (1) \\ \sigma^2_{k+1} &= a_0 + a_1 X_k^2 + b_1 \sigma_k^2 && (2) \end{aligned}$$

where Z_k, $k \in \mathbb{N}$, are i.i.d. and standard normally distributed, and a_0,a_1 and b_1 are parameters, which should be estimated.

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It is simple to simulate from this model.

However analytical computation of VaR and CVaR over a certain time interval consisting of many periods is cumbersome! Check it out!

Chapter 3: Extreme value theory

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
- $\bar{F} := 1 - F$ is called the *right tail* of the univariate distribution function F.

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Terminology: We say a r.v. X has *fat tails* or is *heavy tailed* (h.t.) iff $\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} = \infty$, $\forall \lambda > 0$.

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These two "definitions" are not equivalent!

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has a regular variation with index* $\rho \in \mathbb{R}$ *towards* $+\infty$ *iff*

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If $\rho < 0$*, then the convergence in (3) uniform in every interval* $(b,+\infty)$ *for* $b > 0$*.*

Example

Show that $L \in RV_0$ *holds for the functions L as below:*

(a) $\lim_{x\to+\infty} L(x) = c \in (0,+\infty)$

(b) $L(x) := \ln(1+x)$

(c) $L(x) := \ln(1+\ln(1+x))$

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The converse is not true!