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Definition: (The generalized extreme value distribution (gevd))
Let the distribution function H_γ be given as follows:

$$H_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{wenn } \gamma \neq 0 \\ \exp\{-\exp\{-x\}\} & \text{wenn } \gamma = 0 \end{cases}$$

where $1 + \gamma x > 0$, i.e. the definition area of H_γ is given as

$$\begin{array}{ll} x > -\gamma^{-1} & \text{wenn } \gamma > 0 \\ x < -\gamma^{-1} & \text{wenn } \gamma < 0 \\ x \in \mathbb{R} & \text{wenn } \gamma = 0 \end{array}$$

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H_γ is called *generalized extreme value distribution (gevd)*.
Theorem: (Characterisation of $MDA(H_\gamma)$)

- $F \in MDA(H_\gamma)$ with $\gamma > 0 \iff F \in MDA(\Phi_\alpha)$ with $\alpha = 1/\gamma > 0$.
- $F \in MDA(H_0) \iff F \in MDA(\Lambda)$.
- $F \in MDA(H_\gamma)$ with $\gamma < 0 \iff F \in MDA(\Psi_\alpha)$ with $\alpha = -1/\gamma > 0$.

MDAs: Examples and Characterisations

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Observation: $\lim_{x \to +\infty} \frac{\bar{\Phi}_\alpha(x)}{x^{-\alpha}} = 1$, $\forall \alpha > 0$. Thus for $\Phi_\alpha \in MDA(\Phi_\alpha)$ we have $\bar{\Phi}_\alpha \in RV_{-\alpha}$. Does this generally hold for members of $MDA(\Phi_\alpha)$?

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Theorem: ($MDA(\Phi_\alpha)$, Gnedenko 1943)
$F \in MDA(\Phi_\alpha)$ $(\alpha > 0) \Longleftrightarrow \bar{F} \in RV_{-\alpha}$ $(\alpha > 0)$.
If $F \in MDA(\Phi_\alpha)$, then $\lim_{n \to \infty} a_n^{-1} M_n = \Phi_\alpha$ with $a_n = F^{\leftarrow}(1 - n^{-1})$.

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Examples: The following distributions belong to $MDA(\Phi_\alpha)$:

- Pareto: $F(x) = 1 - x^{-\alpha}$, $x > 1$, $\alpha > 0$.
- Cauchy: $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.
- Student: $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.
- Loggamma: $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} x^{-\alpha-1}$, $x > 1$, $\alpha, \beta > 0$.

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$F \in MDA(\Psi_\alpha)$ $(\alpha > 0) \iff x_F := \sup\{x \in \mathbb{R} \colon F(x) < 1\} < \infty$ and $\bar{F}(x_F - x^{-1}) \in RV_{-\alpha}$ $(\alpha > 0)$.

If $F \in MDA(\Psi_\alpha)$, then $\lim_{n\to\infty} a_n^{-1}(M_n - x_F) = \Psi_\alpha$ with $a_n = x_F - F^{\leftarrow}(1 - n^{-1})$.

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Example: Let $X \sim U(0,1)$. it holds $X \in MDA(\Psi_1)$ with $a_n = 1/n$, $n \in \mathbb{N}$.

Characterisations of MDAs (contd.)

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Observation: $\lim_{x \to +\infty} \frac{\bar{\Lambda}(x)}{e^{-x}} = 1$, $\forall \alpha > 0$.
Thus for $\Lambda \in MDA(\Lambda)$ we have $\bar{\Lambda} \sim e^{-x}$. Does this (or smth. similar) generally hold for members of $MDA(\Lambda)$?

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Theorem: $(MDA(\Lambda))$
Let F be a distribution function with right endpoint $x_F \leq \infty$.
$F \in MDA(\Lambda)$ holds iff there exists a $z < x_F$ such that F can be represented as

$$\bar{F}(x) = c(x) exp\left\{ -\int_z^x \frac{g(t)}{a(t)} dt \right\}, \forall x, z < x \leq x_F,$$

where the functions $c(x)$ and $g(x)$ fulfill $\lim_{x \uparrow x_F} c(x) = c > 0$ and $\lim_{t \uparrow x_F} g(t) = 1$, and $a(t)$ is a positive absolutely continuous function with $\lim_{t \uparrow x_F} a'(t) = 0$.

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See the book by Embrechts et al. for the proofs of the above theorem and of the following theorem concerning the characterisation of $MDA(\Lambda)$.

Characterisations of MDAs (contd.)

Theorem: ($MDA(\Lambda)$, alternative characterisation)
A distribution function F belongs to $MDA(\Lambda)$ iff there exists a a positive function $\tilde{a}$ such that

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x + u\tilde{a}(x))}{\bar{F}(x)} = e^{-u}, \forall u \in \mathbb{R}$$

A possible choice for $\tilde{a}$ is $\tilde{a}(x) = a(x)$ with $a(x) := \int_x^{x_F} \frac{\bar{F}(t)}{\bar{F}(x)} dt$.

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Definition: The function $a(x)$ above is called *mean excess function* and it can be alternatively represented as

$$a(x) := E(X - x | X > x), \forall x \leq x_F .$$

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Examples: The following distributions belong to $MDA(\Lambda)$:

- Normal: $F(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$, $x \in \mathbb{R}$.
- Exponential: $f(x) = \lambda^{-1} \exp\{-\lambda x\}$, $x > 0$, $\lambda > 0$.
- Lognormal: $f(x) = (2\pi x^2)^{-1/2} \exp\{-(\ln x)^2/2\}$, $x > 0$.
- Gamma: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-\beta x\}$, $x > 0$, $\alpha, \beta > 0$.

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  Question: How to check whether this assumption holds?

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  Let $x_{n,n} \leq x_{n-1,n} \leq \ldots \leq x_{1,n}$ be a sorted sample of X_1, X_2,..., X_n.
  qq-plot: $\{(x_{k,n}, F^{\leftarrow}(\frac{n-k+1}{n+1})) \colon k = 1, 2, \ldots, n\}$.

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  Rule of thumb: the larger the quantile the heavier the tails of the distribution!

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Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v. with distribution function F, such that $\bar{F} \in RV_{-\alpha}$, $\alpha > 0$, i.e. $\bar{F}(x) = x^{-\alpha} L(x)$ with $L \in RV_0$.

Goal: Estimate α!

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Goal: Estimate α!

Theorem: (Theorem of Karamata)
Let L be a slowly varying locally bounded function on $[x_0, +\infty)$ for some $x_0 \in \mathbb{R}$. Then the following holds:

(a) For $\kappa > -1$: $\int_{x_o}^{x} t^{\kappa} L(t) dt \sim K(x_0) + \frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$, where $K(x_0)$ is a constant depending on x_0.

(b) For $\kappa < -1$: $\int_{x}^{+\infty} t^{\kappa} L(t) dt \sim -\frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$.

Proof in Bingham et al. 1987.

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The theorem of Karamata implies: $E\left(\ln(X) - \ln(u) \middle| \ln(X) > \ln(u)\right) =$

$$\lim_{u \to \infty} \frac{1}{\bar{F}(u)} \int_u^\infty (\ln x - \ln u) dF(x) = \alpha^{-1}. \qquad (1)$$

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For the empirical distribution $F_n(x) = \frac{1}{n}\sum_{k=1}^{n} I_{[x_k,\infty)}(x)$ and a large threshold $x_{k,n}$ depending on the sample $x_{n,n} \leq x_{n-1,n} \leq \ldots \leq x_{1,n}$ we get:

$$E\left(\ln(X) - \ln(x_{k,n}) | \ln(X) > \ln(x_{k,n})\right) \approx$$

$$\frac{1}{\bar{F}_n(x_{k,n})} \int_{X_{k,n}}^\infty (\ln x - \ln x_{k,n}) dF_n(x) = \frac{1}{k-1} \sum_{j=1}^{k-1} (\ln x_{j,n} - \ln x_{k,n}).$$

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If $k = k(n) \to \infty$ and $k/n \to 0$, then $x_{k,n} \to \infty$ for $n \to \infty$, and (1) implies:

$$\lim_{n\to\infty} \frac{1}{k-1}\sum_{j=1}^{k-1} (\ln x_{j,n} - \ln x_{k,n}) \stackrel{d}{=} \alpha^{-1}$$