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Theorem: (Embrechts et al., 2002)
Let M be the set of returns of the portfolii in $\mathcal{P} := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^{d} |w_i| = 1\}$. Let the asset returns $X = (X_1, X_2, \ldots, X_d)$ be elliptically distributed, $X = (X_1, X_2, \ldots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d \times d}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$. Then VaR_α ist coherent in M, for any $\alpha \in (0.5, 1)$.

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Let $X = (X_1, X_2, \ldots, X_d) = \mu + AY$ be elliptically distributed with $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times k}$ and a spherically distributed vector $Y \sim S_k(\psi)$. Assume that $0 < E(X_k^2) < \infty$ holds $\forall k$. If the risk measure ρ has the properties (C1) and (C3) and $\rho(Y_1) > 0$ for the first component Y_1 of Y, then

$$\arg\min\{\rho(Z(w))\colon w \in \mathcal{P}_m\} = \arg\min\{var(Z(w))\colon w \in \mathcal{P}_m\}$$

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Equivalently, a copula C is a function $C\colon [0,1]^d \to [0,1]$, with the following properties:

1. $C(u_1, u_2, \ldots, u_d)$ is mon. increasing in each variable u_i, $1 \le i \le d$.
2. $C(1,1,\ldots,1,u_k,1,\ldots,1) = u_k$ for any $k \in \{1,\ldots,d\}$ and $\forall u_k \in [0,1]$.
3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1,2,\ldots,d\}$:

$$\sum_{k_1=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$.

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Remark: The k-dimensional marginal distributions of a d-dimensional copula are k-dimensional copulas, for all $2 \le k \le d$.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

1. $h^{\leftarrow}$ is eine monotone increasing left continuous function.

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Lemma: Let X be a r.v. with continuous distribution function F. Then $P\left(F^{\leftarrow}(F(x)) = x\right) = 1$, i.e. $F^{\leftarrow}(F(X)) \overset{a.s.}{=} X$

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Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

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Theorem: (Sklar, 1959)
Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. F_1,…,F_d. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

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Vice-versa, if C is a copula and F_1,…,F_d are d.f., then the function F defined by the equality above is a c.d.f. with marginal d.f. F_1,…,F_d.

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C as above is called *the copula of F*. For a random vector $X \in \mathbb{R}^d$ with c.d.f. F we say that C is *the copula of X*.