Risk theory and risk management in actuarial science
winter term 2018/19

1st work sheet

1. Let $L \sim N(\mu, \sigma^2)$. Show that $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma\Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0,1)$. Further show that $CVaR_\alpha(X) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds, where ϕ is the density function of X as above, and derive also a formula for $CVaR_\alpha(L)$.

2. Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

   (a) Compute $VaR_{0.99}(L_1)$.

   (b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^\Delta)$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^Δ is the linearization of the above mentioned 100-day PF-portfolio loss. Compare the two values and comment on the results.

   Hint: Use the equality $\Phi^{-1}(0.99) \approx 2.3$, where Φ is the distribution function of a random variable $X \sim N(0,1)$.

3. (a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

   (b) Let the distribution function F_L of the loss function L be given by $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and some parameter $\gamma \in (0,1)$ (this is the generalized Pareto distribution). Compute $CVaR_\alpha(L)$.

4. Let the loss L be distributed according to the Students t-distribution with $\nu > 1$ degrees of freedom. The density function of L is given as

   $$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

   Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where t_ν is the distribution function of L.

5. Suppose we have an i.i.d. sample $x_1, x_2, \ldots, x_n$ from a common unknown continuous distribution function F and that we want a confidence interval for $q_{0.8}(F)$ with confidence level $p' \geq p = 0.75$. Use the alternative method without bootstrapping (cf. lecture) to determine the indices of those data points from the (appropriately sorted) sample which could be the boundaries of the required confidence interval.

6. Investigate whether the following functions are slowly varying:

   (a) $L(x) := \ln(1 + \ln(1+x))$

   (b) $f(x) := 3 + \sin x$

   (c) $f(x) := \ln(e + x) + \sin x$

7. Show that the function $L(x) := \exp\{(\ln(1+x))^2 \cos((\ln(1+x))^{1/2})\}$ is slowly varying and has infinite oscillations at infinity in the sense that the following equalities holds:

   $$\lim\inf_{x\to\infty} L(x) = 0 \text{ and } \lim\sup_{x\to\infty} L(x) = \infty\,.$$

8. Show that the following distributions are regularly varying:

(a) The Pareto distribution G_α with parameter α given as $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$, where $\alpha > 0$. Show that $\bar{G}_\alpha(tx)/\bar{G}_\alpha(t) = x^{-\alpha}$ holds for $t > 0$, $x > 0$, thus $\bar{G}_\alpha \in RV_{-\alpha}$.

(b) The Fréchet distribution Φ_α with parameter α given as $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, where $\alpha > 0$. Show that $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

9. Let X and Y be positive random variables representing losses in two lines of business (e.g. losses due to fire and car accidents) of an insurance company. Suppose that X has distribution function F which satisfies $\bar{F} \in RV_{-\alpha}$ for $\alpha > 0$. Moreover suppose that Y has finite moments of all orders, i.e. $E(Y^k) < \infty$, for every $k > 0$. Compute $\lim_{x\to\infty} P(X > x | X + Y > x)$, i.e. the asymptotic probability of a large loss in the fire insurance line given a large total loss.

10. Show that for any distribution F, $F \in DA(G_2)$ if and only if $F \in DA(\Phi)$, where Φ is the standard normal distribution, $\Phi \sim N(0, 1)$, DA stands for "Domain of attraction", and G_2 is the stable distribution with form parameter $\alpha = 2$ (and arbitrary parameters β and c).

    Hint: Apply the *Convergence to types theorem*.

11. Prove the following characterization of the maximum domain of attraction of an extreme value distribution H (also fomulated in the lecture):
    $F \in MDA(H)$ with normalizing and centralizing constants $a_n > 0$, b_n, $n \in \mathbb{N}$, respectively, iff $\lim_{n\to\infty} n\bar{F}_n(a_n x + b) = -\ln(H(x))$, for all $x \in \mathbb{R}$.

12. (The Maxima of the Poisson distribution)
    Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, for some parameter $\lambda > 0$. Show that there exists no extreme value distribution Z such that $X \in MDA(Z)$.

    Hint: Use Leadbetter et al.'s Lemma as follows (cf. lecture). For any discrete non-negative distribution F with right end $x_F = +\infty$ (i.e. a random variable with distribution F can take arbitrarily large values), the following two statements are equivalent for every $\tau \in (0, \infty)$: a) there exists a sequence $u_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} n\bar{F}(u_n) = \tau$, and b) $\lim_{n\to\infty} \frac{\bar{F}(n)}{\bar{F}(n-1)} = 1$.

    You don't need to prove the lemma.