Risk theory and risk management in actuarial science
Winter term 2018/19

3d work sheet

21. Consider the logarithmic daily returns of the close prises of the Nasdaq Composite index (`^IXIC`) and apply the method of the Hill estimator to analyse their tails. Perform the following steps with three different time intervals: (I) from November 1, 1996 till November 2, 2018, (II) from November 1, 1996 till December 26, 2008, and (III) from December 29, 2008 till November 2, 2018. Compare the obtained results and comment upon your findings.

    (a) Compare the tails of the empirical distribution of the data set to the tails of the exponential ditribution by means of the QQ-plot.

    (b) Compute the Hill estimator for the empirical data. Argue carefully upon your choice of the threshold parameter k based on the inspection of the Hill plot as in the case of the fire insurance example discussed in the lecture.

    (c) Based on the Hill estimator give an estimator for the $\mathrm{VaR}_{0.95}$ and the $\mathrm{VaR}_{0.99}$ of the data set.

    The data can be downloaded from `finance.yahoo.com`: search for the required index (you can well search for the abbreviation given in paranthesis above), click 'Historical Data', update the 'Time Period' and 'Frequency' appropriately, and finally klick on 'Download data'.

22. Use the peaks over threshhold (POT) method to analyse the tails of the data described in Exercise 21.

    (a) Argue carefully upon your choice of the threshold parameter k based on the inspection of the plot of the empirical mean excess function (analogously to the case of the fire insurance example discussed in the lecture).

    (b) Maximize the log-likelihood function to obtain estimators for γ and β by using a solver of your choice. Consider the plot of the different values of the estimator $\hat{\gamma}$ of γ in dependence of the threshold parameter k to back your choice for a suitable interval of values of k (cf. the fire insurance example from the lecture).

    (c) Compute estimators for $\mathrm{VaR}_{0.95}$ and $\mathrm{CVaR}_{0.95}$ for the whole interval of reasonable values of k determined in (b). Visualize the dependence of these estimators on k graphically and revise you choice for the interval of values of k, if appropriate.

    (d) Choose a value of k and visualize in one plot the empirical tail distribution and the tail distribution obtained by the POT method. Comment upon your results.

23. Let the random variables X_i, $i = 1, 2$, be such that $X_1 \sim Exp(\lambda)$ and $X_2 = t(X_1)$, where $Exp(\lambda)$ is the exponential distribution with parameter λ and $t\colon \mathbb{R} \to \mathbb{R}$, $t(x) = x^2$. Determine the coefficients of the lower and the upper tail dependence $\lambda_L(X_1, X_2)$, $\lambda_U(X_1, X_2)$, respectively, and conclude that X_1 and X_2 have both a lower and an upper tail dependence. Compute also the coefficient of the linear correlation $\rho_L(X_1, X_2)$, compare the three computed dependence measures and comment on your results.

24. (A coherent premium principle)

    Consider two constants $p > 1$ and $\alpha \in [0, 1)$. Let $(\Omega, \mathcal{F}, P)$ be some fixed probability space and $\mathcal{M}$ be the set of all random variables L on $(\Omega, \mathcal{F})$ for which $E(|L|^p)^{1/p}$ is finite, i.e. $E(|L|^p)^{1/p} < \infty$. Define a risk measure $\rho_{\alpha,p} := E(L) + \alpha(||(L - E(L))^+||_p$ on $\mathcal{M}$, where $||X||_p := E(|X|^p)^{1/p}$ is the L^p-norm of the positive part of the centered random variable $X - E(X)$ for any random variable $X \in \mathcal{M}$. Show that $\rho_{\alpha,p}$ is a coherent risk measure for any $p > 1$ and any $\alpha \in [0, 1)$. So we get a whole family of coherent risk measures $\rho_{\alpha,p}$ for $p > 1$ and $\alpha \in [0, 1)$. How do the parameters α and p influence $\rho_{\alpha,p}$? Which parameter values lead to more "conservative" risk measures?

25. (Generalized scenarios as coherent risk measures)

Denote by $\mathcal{P}$ a set of probability measures on some underlying measurable space $(\Omega, \mathcal{F})$ and set

$$\mathcal{M}_{\mathcal{P}} := \{L\colon L \text{ is a r.v. on } (\Omega, \mathcal{F}),\ E^{Q}(|L|) < \infty \text{ for all } Q \in \mathcal{P}\},$$

where $E^{Q}(X)$ denotes the expected value of a random variable X under the probability measure Q. Then *the risk measure induced by the set of generalized scenarios* $\mathcal{P}$ is the mapping $\rho_{\mathcal{P}}\colon \mathcal{M}_{\mathcal{P}} \to \mathbb{R}$ such that $\rho_{\mathcal{P}}(L) := \sup\{E^{Q}(L)\colon Q \in \mathcal{P}\}$. Show that $\rho_{\mathcal{P}}$ is coherent on $\mathcal{M}_{\mathcal{P}}$ for any set $\mathcal{P}$ of probability measures on $\mathcal{M}_{\mathcal{P}}$. Interprete the scenario based risk measures (cf. lecture) as a risk measure generalized by an appropriately defined set of probability measures on appropriately defined discrete probability spaces[1].

26. (a) Show that $W_d(u_1, u_2, \ldots, u_d == \max\{\sum_{i=1}^{d} u_i - d + 1, 0\}$ is indeed a lower bound for any copula $C\colon [0,1]^d \to [0,1]$, i.e. that $W_d(u_1, u_2, \ldots, u_d) \le C(u_1, u_2, \ldots, u_d)$ holds for any $d \in \mathbb{N}$, $d \ge 2$, any $(u_1, \ldots, u_d) \in [0,1]^d$ and any copula C as above.

    (b) Show that the Fréchet lower bound W_d is not a copula for $d \ge 3$.

    Hint: Show that the rectangle inequality

    $$\sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0,$$

    where $(a_1, a_2, \ldots, a_d)$, $(b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$ and $u_{k1} = a_k$ und $u_{k2} = b_k$ for all $k \in \{1, 2, \ldots, d\}$, is violated if $d \ge 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.

27. Let X_i, $i = 1, 2$, be two lognormally distributed random variables with $X_1 \sim \mathit{Lognormal}(0,1)$ und $X_2 \sim \mathit{Lognormal}(0, \sigma^2)$, $\sigma > 0$. Compute $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ in dependence of σ and compare their values for different values of $\sigma > 0$. What can you say about the copula of (X_1, X_2) in each of the cases? Plot the graphs of $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ as functions of σ and comment on the behaviour of these functions for $\sigma \to +\infty$?

    Hint: Consider $X_1 = \exp(Z)$ and $X_2 = \exp(\sigma Z)$ or $X_2 = \exp(-\sigma Z)$ for a standard normally distributed random variable Z.

[1] It can be shown that in the case of discrete probability spaces any coherent risk measure is induced by some set of generalized scenarios as described above, see Proposition 6.11 in *A.J. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management: Concepts, Techniques amd Tools , Princeton Unversity Press, 2005.*