Risk theory and risk management in actuarial science
Winter term 2018/19

4th work sheet

28. Construct two random vectors $(X_1, X_2)^T$ and $(Y_1, Y_2)^T$ with different joint distributions $F_{(X_1,X_2)}$, $F_{(Y_1,Y_2)}$, respectively, such that

    (a) the variables X_1, X_2, Y_1, Y_2 are standard normally distributed, i.e. $X_1, X_2, Y_1, Y_2 \sim N(0, 1)$,

    (b) the two X-variables and the two Y-variables are uncorrelated, respectively, i.e. $\rho_L(X_1, X_2) = 0$, $\rho_L(Y_1, Y_2) = 0$, and

    (c) the α-quantiles of the corresponding sums are different, i.e. $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds for some $\alpha \in (0, 1)$, where $F_{X_1+X_2}$, $F_{Y_1+Y_2}$ are the distributions of X_1+X_2 and Y_1+Y_2, respectively.

    Conclude that in general it is not possible to draw conclusions about the loss of a portfolio if only the loss distributions of the single assets in portfolio and their mutual linear correlation coefficients are known.

    Hint: Choose (X_1, X_2) to be bivariate standard normally distributed, i.e. $(X_1, X_2) \sim N_2(0, I_2)$, where 0 denotes the zero vector in $\mathbb{R}^2$ and I_2 denotes the identity matrix in $\mathbb{R}^{2\times 2}$. Choose Y_1 to be standard normally distributed, $Y_1 \sim N(0, 1)$, and set $Y_2 := VY_1$, where V is a discrete random variable independent on Y_1 with values 1 and -1 taken with probability 1/2 each.

29. (Co-monotonicity and anti-monotonicity)

    (a) Let Z be a random variable with continuous cumulative distribution function F, $Z \sim F$. Let f_1, f_2 be to monotone increasing functions on $\mathbb{R}$ and let f_3 be a monotone decreasing function on $\mathbb{R}$. Let $X_i = f_i(Z)$, for $i = 1, 2, 3$. Show that the Fréchet upper bound M is a copula of (X_1, X_2) and the Fréchet lower bound W is a copula of (X_1, X_3).

    (b) Let W be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$.

    (c) Let M be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$.

30. Prove the following equality for the rank correlation Spearman's rank correlation coefficient of a random vector $(X_1, X_2)^T$ with continuous marginal distributions and unique copula C:

    $\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1u_2)du_1du_2 = 12 \int_0^1 \int_0^1 C(u_1, u_2)du_1du_2 - 3.$

31. The Gumbel family C_θ^{Gu} and the Clayton family C_θ^{Cl} are two one-parametric families of copulas given as

    $$C_\theta^{\mathrm{Gu}}(u_1, u_2) := \exp\left(- \left[(-\ln u_1)^\theta + (-\ln u_2)^\theta \right]^{1/\theta} \right), \ \theta \geq 1, \text{ and}$$

    $$C_\theta^{\mathrm{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \ \theta > 0.$$

    (a) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependence for the copulas C_θ^{Gu}, C_θ^{Cl}, respectively.

    (b) The independence copula Π is given by $\Pi(u_1, u_2) := u_1u_2$, for $(u_1, u_2) \in [0, 1]^2$. Show that C_θ^{Gu} tends to the independence copula Π if θ tends to 1 and to the upper Fréchet bound M if θ tends to infinity. In this case we say that *the lower limit of the Gumbel copula is the independence copula* Π *and its upper limit is the Fréchet upper bound* M. Analogously show that the lower limit of the Clayton copula is the independence copula Π for $\theta \to 0^+$ and its upper limit is the Fréchet upper bound M for $\theta \to +\infty$. Now considerer an extension of the Clayton copula C_θ^{Cl} for $\theta \in [-1, 0)$, defined as an Archimedian copula with generator $\phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ for $t \in (0, 1]$ and $\phi_\theta(0) = +\infty$. Show that for $\theta = -1$ the Clayton copula C_{-1}^{Cl} coincides with the Fréchet lower bound W.

32. (a) Let $(X_1, X_2)^T$ be a t-distributed random vektor with ν degrees of freedom, expected value $(0,0)$ and linear correlation coefficient matrix $\rho \in (-1,1]$, i.e. $(X_1, X_2)^T \sim t_2(\vec{0}, \nu, R)$ where R is 2×2 matrix with 1 on the diagonal and ρ outside the diagonal. Show that the following equality holds for $\rho > -1$:

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$$

Hint: Use the fact (no need to prove it!) that conditional on $X_1 = x$ the following holds

$$\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}\,.$$

Recall the stochatic representation of the bivariate t-distribution as $\mu + \sqrt{W}AZ$, where Z is bivariate standard normally distributed and W is such that $\frac{\nu}{W} \sim \chi^2_\nu$ while being independent on Z (cf. lecture).

(b) Apply (a) to conclude that for a random vector with continuous marginal distributions $(X_1, X_2)^T$ and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and a correlation matrix R as in (a) the following equalities holds:

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2t_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\,.$$

Archimedian Copulas

33. (a) Show that for every $\theta \in \mathbb{R} \setminus \{0\}$ the function $\phi^{Fr}_\theta(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ generates an Archmedian copula, the so-called Frank copula $C^{Fr}_\theta \colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C^{Fr}_\theta(u_1, u_2) = -\frac{1}{\theta}\ln\left(1 + \frac{(\exp(-\theta u_1)-1)(\exp(-\theta u_2)-1)}{\exp(-\theta)-1}\right), \theta \in \mathbb{R} \setminus \{0\}\,.$$

(b) Show that for every $\theta > 0$ and for every $\delta \geq 1$ the function $\phi^{GC}_{\theta,\delta}(t) = \theta^{-\delta}(t^{-\theta}-1)^\delta$ generates an Archmedian copula, the so-called *generalized Clayton copula* $C^{GC}_{\theta,\delta} \colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C^{GC}_{\theta,\delta}(u_1, u_2) = \{[(u_1^{-\theta}-1)^\delta + (u_2^{-\theta}-1)^\delta]^{1/\delta} + 1\}^{-1/\theta},\ \theta \geq 0\,, \delta \geq 1\,.$$

(c) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependency for the copulas C^{Fr}_θ and $C^{GC}_{\theta,\delta}$, respectively.