**Risk theory and risk management in actuarial science**
**Winter term 2018/19**
**5th work sheet**

34. **Asymmetric bivariate copulas**
Let $C_\theta$ be any exchangeable bivariate copula. Then a parametric family of asymmetric copulas $C_{\theta,\alpha,\beta}$ is obtained by setting

$$C_{\theta,\alpha,\beta}(u_1, u_2) = u_1^{1-\alpha} u_2^{1-\beta} C_\theta(u_1^\alpha, u_2^\beta), \ 0 \le u_1, u_2 \le 1, \tag{1}$$

where $0 \le \alpha, \beta \le 1$. When $C_\theta$ is an Archimedian copula we refer to the copulas constructed by (1) as asymmetric bivariate Archimedian copulas.

Check that $C_{\theta,\alpha,\beta}$ defined as above is a copula by constructing a random vector with distribution function $C_{\theta,\alpha,\beta}$ and observing that its margins are standard uniform on $[0,1]$. Show that such a random vector $(U_1, U_2)$ can be generated as follows:

(a) Generate a random pair $(V_1, V_2)$ with distribution function $C_\theta$.
(b) Generate, independently of $V_1$, $V_2$, two independent standard uniform variables $\bar{U}_1$ and $\bar{U}_2$.
(c) Set $U_1 = \max\{V_1^{1/\alpha}, \bar{U}_1^{1/(1-\alpha)}\}$ and $U_2 = \max\{V_2^{1/\beta}, \bar{U}_2^{1/(1-\beta)}\}$

Show that $C_{\theta,\alpha,\beta}$ is not exchangeable in general. What conditions should fulfill its parameters so as to obtain an exchangeable $C_{\theta,\alpha,\beta}$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 0$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 1$?

35. **Three-dimensional non-exchangeable Archimedian copulas**
Suppose that $\phi_1$ and $\phi_2$ are two *strict* generators of Archimedian copulas, i.e. $\phi_1$ and $\phi_2$ are completely monotonic decreasing functions for which the pseudo-inverse function coincides with the ordinary inverse function. Consider

$$C(u_1, u_2, u_3) = \phi_2^{-1}(\phi_2 \circ \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3)). \tag{2}$$

If $\phi_1^{-1}$, $\phi_2^{-1}$ and $\phi_2 \circ \phi_1^{-1}$ are completely monotonic decreasing functions mapping $[0, \infty]$ to $[0, \infty]$, then $C$ defined by (2) is a copula.

Let $(U_1, U_2, U_3)$ be a random vector with distribution function $C$ as defined by (2).

(a) Show that if $\phi_1 \neq \phi_2$, then only $U_1$ and $U_2$ are exchangeable, i.e. $(U_1, U_2, U_3) \overset{d}{=} (U_2, U_1, U_3)$, but no other swapping of subscritps is possible. Show moreoever that if $\phi_1 = \phi_2$, than $C$ is an exchangeable copula.
(b) Show that all bivariate margins of $C$ defined by (2) are themselves Archimedian copulas; the margins $C_{13}$ (obtained by components 1 and 3) and $C_{23}$ (obtained by components 2 and 3) have generator $\phi_2$, whereas the margin $C_{12}$ (obtained by components 1 and 2) has generator $\phi_1$.

36. **Equivalent threshold models**
Let $X = (X_1, X_2, \ldots, X_m)'$ be an $m$-dimensional random vector and let $D \in \mathbb{R}^{m \times n}$ be a deterministic matrix with elements $d_{ij}$ such that for every $i$, $1 \le i \le m$, the elements of the $i$-th row form a set of increasing thresholds satisfying $d_{i,1} < d_{i2} \ldots < d_{in}$. Introduce additionally $d_{i0} = -\infty$, $d_{i,n+1} = +\infty$ and set

$$S_i = j \iff d_{ij} < X_i \le d_{i,j+1}, \text{ for } j \in \{0, \ldots, n\}, i \in \{1, \ldots, m\}.$$

Then $(X, D)$ is said to define a *threshold model* for the *state vector* $S = (S_1, \ldots, S_m)'$. We refere to $X$ as the *vector of critical variables* and denote its marginal distribution functions by $F_i(x) = P(X_i \le x)$, for $i \in \{1, 2, \ldots, m\}$. The $i$-th row of $D$ contains the critical thresholds for firm $i$. By definition, default (corresponding to event $S_i = 0$) occurs iff $X_i \le d_{i1}$, thus the default probability of

company $i$ is given by $\bar{p}_i := F_i(d_{i1})$. Let $Y_i$ be the default indicator of company $i$, i.e. $Y_i \in \{0,1\}$ with $Y_i = 1$ iff company 1 defaults, hence $Prob(Y_i = 1) = \bar{p}_i$ and $Prob(Y_i = 0) = 1 - \bar{p}_i$, for $1 \le i \le m$. We denote by $\rho(Y_i, Y_j)$ the *default correlation* of two firms $i \ne j$; this quantity depends on $E(Y_i, Y_j)$ (how?) which in turn depends on the joint distribution of $(X_i, X_j)$, and hence on the copula of $(X_1, X_2, \ldots, X_m)'$. (Notice that in general the latter is not fully determined by the *asset correlation* $\rho(X_i, X_j)$.)

Two threshold models $(X, D)$ and $(\tilde{X}, \tilde{D})$ for the state vectors $S$ and $\tilde{S}$, respectively, are called equivalent, iff $S$ and $\tilde{S}$ have the same probability distribution.

Show that two threshold models $(X, D)$ and $(\tilde{X}, \tilde{D})$ with state vectors $S$ and $\tilde{S}$, respectively, are equivalent if the following conditions hold:

(a) The marginal distributions of the random vectors $S$ and $\tilde{S}$ coincide, i.e. $P(S_i = j) = P(\tilde{S}_i = j)$, for all $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$.

(b) $X$ and $\tilde{X}$ admit the same copula $C$.

37. A bank has a loan portfolio of 100 loans. Let $X_k$ be the default indicator for loan $k$ such that $X_k = 1$ in case of default and 0 otherwise, for $k \in \{1, \ldots, 100\}$.

(a) Supoose that $X_k$ are independent and identically distributed with $P(X_k = 1) = 0.01$. Compute the expected value $E(N)$ of the number $N$ of defaults and $P(N = k)$ for $k \in \{0, 1, \ldots, 100\}$.

(b) Consider the risk factor $Z$ which reflects the state of the economy. Suppose that conditional on $Z$ the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z$, where $P(Z = 0.01) = 0.9$ and $P(Z = 0.11) = 0.1$. Compute the expected value $E(N)$ where $N$ is defined as in (a).

(c) Consider the risk factor $Z$ which reflects the state of the economy. Suppose that conditional on $Z$ the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z^9$, where $Z$ is uniformly distributed on $(0, 1)$. Compute the expected value $E(N)$ where $N$ is defined as in (a).

38. An $m$-dimensional random vector $X$ is said to have a $p$-dimensional conditional independence structure with conditioning variable $\Psi$ iff there is some $p \in \mathbb{N}$, $p < m$, and a $p$-dimensional random vector $\Psi = (\Psi_1, \ldots, \Psi_m)^t$, such that the random variables $X_1, \ldots, X_m$ are independent conditional on $\Psi$. Consider a threshold model $(X, D)$ as defined in Exercise 36 and assume that $X$ has a $p$-dimensional conditional independence structure with conditioning variable $\Psi$. Show that the default indicators $Y_i = \mathbb{I}_{X_i \le d_{i1}}$ follow a Bernoulli mixture model with factor $\Psi$. How are given the conditional default probabilities for this Bernoulli mixture model?

39. Suppose that the critical variables $X = (X_1, \ldots, X_m)^t$ have a normal mean-variance mixture distribution, i.e. $X = m(W) + \sqrt{W}Z$ with an $m$-dimensional random vector $Z$, a positive, scalar random variable $W$ independent of $Z$, and a measurable function $m: [0; +\infty) \to \mathbb{R}^m$. Assume that $Z$ (and hence $X$) follows a linear factor model of the form $Z = BF + \epsilon$, where $F \sim N_p(\vec{0}, \Omega)$ is a $p$-dimensional normally distributed vector with expected value $\vec{0} \in \mathbb{R}^p$ and covariance matrix $\Omega \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{m \times p}$ is a deterministic loading matrix, and the components $\epsilon_1, \ldots, \epsilon_m$ of $\epsilon$ are i.i.d. normally distributed random variables which are also independent of $F$. Show that F hast a $(p+1)$-conditional independence structure (see the definition in Exercise 38). How are given the conditional default probabilities for the corresponding Bernoulli mixture model of the default indicators $Y_i$ in this case (cf. Exercise 38)?

40. (Application of Archimediam copulas in threshold models)

Consider a threshold model $(X, D)$ where $X$ has an Archimedian copula $C$ with generator $\phi$ such that $\phi^{-1}$ is the Laplace transform of some nonnegative distribution function $G$ with $G(0) = 0$. Let $d = (d_{i1}, \ldots, d_{mi})^t$ denote the first column of $D$ containing the default thresholds. We write then $(X, d)$ for a threshold model of default with an Archimedian copula dependence and denote by $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_m)^t$ the vector of default probabilities, where $\bar{p}_i = \mathbb{P}(X_i \le d_{i1})$, for $i \in \{1, 2, \ldots, m\}$.

Consider a nonnegative random variable $\Psi \sim G$ and random variables $U_1,\ldots,U_m$ that are conditionally independent given $\Psi$ with conditional distribution function $\mathbb{P}(U_i \leq u | \Psi = \psi) = \exp(-\psi\phi(u))$, for $u \in [0,1]$. Check that $U = (U_1,\ldots,U_m)^t$ has distribution function $C$. Show that $(X,d)$ and $(U,\bar{p})$ are two equivalent threshold models for default (cf. Exercise 36). How are given the conditional default probabilities $p_i(\Psi)$ in this case?

Consider the Clayton copula $C_\theta^{Cl}$ with generator function $\phi(t) = t^{-\theta} - 1$ and assume that we want to construct a Bernoulli mixture model that is equivalent to a threshold model driven by $C_\theta^{Cl}$. In this Bernoulli mixture model all conditional default probabilites $p_i(\Psi)$ would coincide; such a Bernoulli mixture model is called *exchangeable*. Assume moreover that the probability of default for any creditor is given by $\pi$ and the probability that an arbitrary pair of creditors defaults is given by $\pi'$. What value of $\theta$ would lead to the required exchangeable Bernoulli mixture modell?