HOPF ALGEBRAS IN COMBINATORICS
TU GRAZ, SUMMER 2024

Exercise Set 1

Definition. Let $f : V \to W$ be a linear map. The *image of* f is the vector subspace of W given by $\mathrm{Im}(f) := \{f(v) \,:\, v \in V\}$. The *kernel of* f is the vector subspace of V given by $\mathrm{Ker}(f) := \{v \in V \,:\, f(v) = 0\}$.

Exercise 1. Let $f : V \to V'$ and $g : W \to W'$ be two linear maps. Show that $\mathrm{Im}(f \otimes g) = \mathrm{Im}(f) \otimes \mathrm{Im}(g)$ and $\mathrm{Ker}(f \otimes g) = V \otimes \mathrm{Ker}(g) + \mathrm{Ker}(f) \otimes W$.

Exercise 2. Let U, V be vector spaces and $V' \subseteq V$ be a subspace. Show that $U \otimes (V/V') \cong (U \otimes V)/(U \otimes V')$.

Exercise 3. Let V and W be two vector spaces.

(1) If $\sum_{i=1}^{n} v_i \otimes w_i = 0$ in $V \otimes W$ and $\{v_1, \ldots, v_n\}$ are linearly independent, show that $w_i = 0$, for any $1 \leq i \leq n$.
(2) Let $W' \subseteq W$ be a subspace. If $\sum_{i=1}^{n} v_i \otimes w_i \in V \otimes W'$ and $\{v_1, \ldots, v_n\}$ are linearly independent, show that $w_i \in W'$, for any $1 \leq i \leq n$.

Exercise 4. Let V, W be two vector spaces, and assume that V has finite dimension. Show that $V^* \otimes W$ is isomorphic to $\mathrm{Hom}(V, W) := \{f : V \to W \,:\, f \text{ is linear}\}$.

Exercise 5. Let X, Y be two indeterminates. Show that $\mathbb{K}[X] \otimes \mathbb{K}[Y] = \mathbb{K}[X, Y]$.

Definition. Let V be a vector space. The *symmetric algebra* $S(V)$ *of* V is the quotient of the tensor algebra $T(V)$ by the ideal generated by the elements $v_1 v_2 - v_2 v_1$, for any $v_1, v_2 \in V$.

Exercise 6. Let V be a vector space and consider its symmetric algebra $S(V)$. Show that

(1) $S(V)$ is commutative.
(2) $S(V)$ satisfies the following universal property: for any linear map $f : V \to A$ where A is a **commutative** algebra, there exists a unique algebra morphism $F : S(V) \to A$ such that $F(v) = f(v)$, for any $v \in V \subseteq S(V)$.
(3) If $\{v_i\}_{i \in I}$ is a basis of V, then the set

$$\left\{ \prod_{i \in I} v_i^{a_i} \,:\, a_i = 0 \text{ for all but finitely many } i \in I \right\}$$

is a basis of $S(V)$. The elements of this basis are called *monomials* on the v_i's.

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Exercise 7. Let $q \in \mathbb{K}$ be non-zero. Consider the algebra A_n with generators $e_1, \dots, e_{n-1}$ subject to the relations

$$\begin{aligned} e_i^2 &= e_i && \text{for all } i, \\ e_i e_j &= e_j e_i && \text{for all } i, j \text{ with } |i-j| > 1, \\ e_i e_j e_i &= q e_i && \text{for all } i, j \text{ with } |i-j| = 1. \end{aligned}$$

Prove that the dimension of A_n is the n-th *Catalan number* $C_n := \frac{1}{n+1}\binom{2n}{n}$.

Exercise 8. Let (C, Δ, ϵ) be a coalgebra and $c \in C$. Show that:

(1) $\sum_{(c)} c_{(1)} \otimes \cdots \otimes \Delta(c_{(i)}) \otimes \cdots \otimes c_{(n)} = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n+1)}$, for any $1 \leq i \leq n$.

(2) $\sum_{(c)} c_{(1)} \otimes \cdots \otimes \epsilon(c_{(i)}) \otimes \cdots \otimes c_{(n)} = \sum_{(c)} c_{(1)} \otimes \cdots \otimes c_{(n-1)}$, for any $1 \leq i \leq n$.

Definition. Let (C, Δ, ϵ) be a coalgebra.

- We say that $c \in C$ is a *group-like* element if $\epsilon(c) = 1$ and $\Delta(c) = c \otimes c$. The set of group-like elements is denoted by $\Gamma(C)$
- We say that $f \in C^*$ is a *character* if f is an algebra morphism. The set of characters is denoted by $\mathrm{Alg}(C^*, \mathbb{K})$.

Exercise 9. Let C be a finite-dimensional coalgebra. Show that there is a set bijection between $\Gamma(C)$ and $\mathrm{Alg}(C^*, \mathbb{K})$.