HOPF ALGEBRAS IN COMBINATORICS
TU GRAZ, SUMMER 2024

Exercise Set 2

Exercise 1. Let (C, Δ, ϵ) be a coalgebra. Prove the following identities:

(1) $\Delta(c) = \sum_{(c)} \epsilon(c_{(2)})\Delta(c_{(1)})$.

(2) $c = \sum_{(c)} \epsilon(c_{(1)})\epsilon(c_{(3)})c_{(2)}$.

Exercise 2. Show that a coalgebra (C, Δ, ϵ) is cocommutative if and only if $\Delta : C \to C \otimes C$ is a morphism of coalgebras.

Exercise 3. Let C be a coalgebra and $c \in C$. Show that c is a group-like element if and only if $\Delta(c) = c \otimes c$ and $c \neq 0$.

Exercise 4. Let S be a set and consider the coalgebra $\mathbb{K}S$ with coproduct given by $\Delta(s) = s \otimes s$ for any $s \in S$ and counit $\epsilon(s) = 1$ for any $s \in S$. Determine all the subcoalgebras of $\mathbb{K}S$.

Exercise 5. Let V, W be two vector spaces. Prove that the bialgebras $S(V) \otimes S(W)$ and $S(V \oplus W)$ are isomorphic.

Exercise 6. Let G and G' be two groups. Prove that $\mathbb{K}G \otimes \mathbb{K}G'$ is isomorphic to a bialgebra $\mathbb{K}G''$ and determine G''.

Exercise 7. Consider the algebra H_4 generated by indeterminates g and x subject to the relations $g^2 = 1$, $x^2 = 0$, and $xg = -gx$. Prove that the coproduct given by $\Delta(g) = g \otimes g$ and

$$\Delta(x) = x \otimes 1 + g \otimes x$$

and the counit given by $\epsilon(g) = 1$ and $\epsilon(x) = 0$ turn H_4 into a non-commutative and non-cocommutative bialgebra.

Exercise 8. Let $C = M_n^*(\mathbb{K})$ be the matrix coalgebra. Let A be any algebra. Show that the convolution algebra $\mathrm{Hom}(C, A) \cong M_n(A)$, where the latter is the algebra of all $n \times n$ matrices with coefficients in A with the usual matrix operations.

Exercise 9 (Slightly more complicated)**.** If two finite posets P and Q have isomorphic incidence coalgebras, prove that they are isomorphic

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