Hopf algebras in Combinatorics

TU Graz - Summer semester 2024

Instructors: Adrián Celestino (celestino@math.tugraz.at) and Yannic Vargas (yvargaslozada@tugraz.at).

Course description: The objective of the course is to study the classical notion of Hopf algebras and relevant applications in Combinatorics. Roughly speaking, a Hopf algebra is a vector space together with two operations, called multiplication and comultiplication which are related in an interesting way. In combinatorics, the comultiplication operation models the notion of breaking down a complex structure into simpler components, mirroring the combinatorial principle of decomposing a counting problem into smaller, more manageable subproblems. The algebraic properties of Hopf algebras have proven to be useful in understanding and solving combinatorial problems, providing a powerful framework for studying the interplay between algebra and combinatorics.

Course website:
https://www.math.tugraz.at/~celestino/HopfAlgebras2024.html

First meeting!

Time: Friday 1 March at 14:00.
Location: Seminarraum A306 (ST03014). Steyrergasse 30/III.

Times and location: To be determined with the course participants.

Level: Master and PhD students.

Prerequisites: Solid foundations in Linear Algebra as well as Groups and Rings.

Tentative list of contents:

- **Introduction.** Tensor products.
- **Algebras and coalgebras.** Diagrammatic definition, examples, duals, fundamental theorem of coalgebras.
- **Bialgebras.** Definition, quotients, primitive space.
- **Convolution product and Hopf algebras.** Definition, properties of the antipode.

- **Graduation and connexity.** Graded bialgebras, connected Hopf algebras, existence of the antipode.
- **Structural theorems.** Poincaré–Birkhoff–Witt Theorem and Cartier–Milnor–Moore Theorem.
- **Characters and infinitesimal characters.** Group of characters of a Hopf algebra, Lie algebra of infinitesimal characters of a Hopf algebra, exponential and logarithm maps.
- **Tree-like Hopf algebras.** Connes–Kreimer Hopf algebra, Grossman–Larson Hopf algebra, duality, extraction-contraction Hopf algebra.
- **Incidence Hopf algebras.** Posets, definition, examples, antipode formula.
- **Combinatorial Hopf algebras/Hopf monoids.** Hopf algebras of symmetric functions Sym; non-commutative symmetric functions NSym, quasi-symmetric functions QSym and free quasi-symmetric functions $\mathfrak{S}\mathsf{Sym}$. QSym as the terminal object in the category of combinatorial Hopf algebras. Analogues structures for Hopf monoids. Generalized permutahedra.

Grading: There will be exercise sessions every other week. Participation in these exercise sessions will count toward the grading. In addition, the students will be required to work on a final project on a selected topic related to the course together with an oral presentation about it.

Main references: We will not follow a textbook for the course. However, the lectures will follow parts of:

[1] Pierre Cartier and Frédéric Patras. *Classical Hopf algebras and their applications.* Vol. 2. Berlin Heidelberg: Springer, 2021.

[2] David E. Radford. *Hopf algebras.* Vol. 49. World Scientific, 2011.

Some secondary references: The following books and articles will be relevant to the course.

[3] Marcelo Aguiar and Federico Ardila. *Hopf monoids and generalized permutahedra.* American Mathematical Society, 2023.

[4] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras.* Vol. 29. Providence, RI: American Mathematical Society, 2010.

[5] Darij Grinberg and Victor Reiner. Hopf algebras in combinatorics. *arXiv preprint* arXiv:1409.8356, 2014.

[6] Michiel Hazewinkel, Nadezhda Mikhaĭlovna Gubareni, and Vladimir V. Kirichenko. *Algebras, rings and modules: Lie algebras and Hopf algebras.* Vol. 3. American Mathematical Society, 2010.

[7] Dominique Manchon. Hopf algebras in renormalisation. *Handbook of algebra,* 5, 365-427, 2008.

[8] Moss E. Sweedler. *Hopf algebras.* Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969.