Hopf Algebras in Combinatorics

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Intuitively, a Hopf algebra is a vector space H with a multiplication $m: H\otimes H \to H$ and a comultiplication $\Delta: H \to H\otimes H$ satisfying several restricted rules:

there is $S: H \to H$ such that

$$\begin{tikzcd} & H\otimes H \arrow[r, "S\otimes id"] & H\otimes H \arrow[dr, "m"] & \\ H \arrow[ur, "\Delta"] \arrow[r, "\epsilon"] \arrow[dr, "\Delta"'] & \mathbb{K} \arrow[r, "\eta"] & & H \\ & H\otimes H \arrow[r, "id\otimes S"'] & H\otimes H \arrow[ur, "m"'] & \end{tikzcd}$$

commutes.

In this course, we will focus on Hopf algebras arising from combinatorial objects such as partitions, permutations, trees, graphs, etc.

1. Tensor Product of vector spaces

Throughout the course, we will only consider vector spaces over a field $\mathbb{K}$ of characteristic 0 (i.e. $\forall\, a\in\mathbb{K}$, there is no $p\in\mathbb{N}$ such that $\underbrace{a+a+\dots+a}_{p \text{ times}}=0$).

Recall that, for any V vector space and $W\subseteq V$ vector subspace, we define the quotient V/W as the set of classes in V under the equivalence relation.

$$x\sim y \iff x-y\in W.$$

The quotient is a vector space: if $\bar{x}\in V/W$ is the class of $x\in V$, define $\bar{x}+\bar{y} := \overline{x+y}$ and $\lambda\bar{x} := \overline{\lambda x}$, for any $x,y\in V$ and $\lambda\in\mathbb{K}$.

Lemma Let X be a set. There exists a vector space $\mathbb{K}X$ with basis X and this space is unique up to isomorphism fixing X.

We can interpret the elements of $\mathbb{K}X$ as formal linear combinations of elements of x:
$$\mathbb{K}X = \mathbb{K}\text{-span}\{X\} = \Big\{\sum_{x\in X}\lambda_x x : \lambda_x\in\mathbb{K} \text{ and } \lambda_x=0 \text{ for all but finitely many } x\in X\Big\}.$$

Definition Let V_1, V_2, W be vector spaces. A map $f: V_1\times V_2\to W$ is bilinear if it is linear in each of its argument when the other is fixed, i.e.
$$f(ax+by,\, a'x'+b'y') = abf(x,y)+ab'f(x,y')+a'bf(x',y)+a'b'f(x',y'), \quad \forall\, x,y\in V_1,\ x',y'\in V_2,\ a,b,a',b'\in\mathbb{K}.$$

Now, take $f: V_1\times V_2\to V$ bilinear. If $h: V\to W$ is a linear map, then $h\circ f: V_1\times V_2\to W$ is a bilinear map. One may ask whether it is possible to choose V and f such that every bilinear map of $V_1\times V_2$ can be obtained in this way. This is called the universal problem for bilinear functions. The next theorem gives us the pair that solves the problem.

Theorem Let V_1 and V_2 be two vector spaces. There exists a pair $(V,\otimes)$ such that V is a vector space and $\otimes: V_1\times V_2 \longrightarrow V$, $(v_1,v_2)\longmapsto v_1\otimes v_2$ is a bilinear map satisfying the following universal property: for any W vector space and $f: V_1\times V_2\to W$ bilinear map, there exists a unique linear map $F: V\to W$ such that the following diagram is commutative:

$$\begin{tikzcd} V_1\times V_2 \arrow[r, "f"] \arrow[d, "\otimes"'] & W \\ V \arrow[ur, "F"'] & \end{tikzcd}$$

The pair $(V,\otimes)$ is unique up to isomorphism.

Proof. Existence: Take $\mathbb{K}(V_1\times V_2)$ the vector space generated by the set $V_1\times V_2$. Also, consider I the vector subspace of $\mathbb{K}(V_1\times V_2)$ given by
$$I := \mathbb{K}\Big\{(v_1+v_1',v_2)-(v_1,v_2)-(v_1',v_2),\ (v_1,v_2+v_2')-(v_1,v_2)-(v_1,v_2'),\ (\lambda v_1,v_2)-\lambda(v_1,v_2),\ (v_1,\lambda v_2)-\lambda(v_1,v_2) \Big\}$$
$$v_1,v_1'\in V_1,\quad v_2,v_2'\in V_2,\quad \lambda\in\mathbb{K}.$$

Then we define $V := \mathbb{K}(V_1 \times V_2)/I$. Also, define the map $\otimes : V_1 \times V_2 \longrightarrow V$ by $(v_1, v_2) \longmapsto v_1 \otimes v_2 := \overline{(v_1, v_2)}$. By definition of I, $\otimes$ is bilinear.

We will show that $(V, \otimes)$ satisfies the universal property. Let W be a vector space and consider a bilinear map $g : V_1 \times V_2 \longrightarrow W$. Notice that we can define a linear map $g' : \mathbb{K}(V_1 \times V_2) \longrightarrow W$ by $g'(v_1, v_2) := g(v_1, v_2)$ ($V_1 \times V_2$ is basis of $\mathbb{K}(V_1 \times V_2)$). Define now $F : V \longrightarrow W$ by $F(\overline{(v_1, v_2)}) := g'(v_1, v_2)$, $\forall (v_1, v_2) \in V_1 \times V_2$. Notice that F is well-defined and linear since $I \subseteq \ker(g') = \{x \in \mathbb{K}(V_1 \times V_2) : g'(x) = 0\}$. (because g is bilinear) By definition, it satisfies that $F \circ \otimes = g$.

It remains to prove that F is unique. Suppose that there is another linear map $F' : V \to W$ such that $F' \circ \otimes = g$. Then

$$F'(\overline{(v_1, v_2)}) = F' \circ \otimes (v_1, v_2) = g(v_1, v_2) = F \circ \otimes (v_1, v_2) = F(\overline{(v_1, v_2)}), \quad \forall (v_1, v_2) \in V_1 \times V_2.$$

Since $\{\overline{(v_1, v_2)}\}$ generates V, we have that $F = F'$ on V.

- Uniqueness: Assume that there is another $(V', \otimes')$. Since $\otimes'$ is bilinear, there exists a unique linear map $\Phi : V \longrightarrow V'$, $v_1 \otimes v_2 \longmapsto v_1 \otimes' v_2$. Analogously, there is a linear map $\Phi' : V' \longrightarrow V$, $v_1 \otimes' v_2 \longmapsto v_1 \otimes v_2$.

  Observe that $\Phi' \circ \Phi : V \longrightarrow V$ satisfies $\Phi' \circ \Phi(v_1 \otimes v_2) = \otimes(v_1, v_2)$. By uniqueness, we have $id_V = \Phi' \circ \Phi$. In the same way, we can show that $id_{V'} = \Phi \circ \Phi'$. Therefore Φ and Φ' are mutually inverse linear maps, i.e. $V \cong V'$ as vector spaces. ∎

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\otimes"] \arrow[d, "\otimes"'] & V \\ V \arrow[ur, "{id,\ \Phi' \circ \Phi}"'] & \end{tikzcd}$$

Definition The pair $(V, \otimes)$ from the previous theorem is called the *tensor product of V_1 and V_2*. It will be denoted by $V_1 \otimes V_2 := V$. The image of (v_1, v_2) through $\otimes$ is denoted by $v_1 \otimes v_2$. In the practice, we can consider $v_1 \otimes v_2$ as a pair with the bilinear property:

$$(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v \in V_1 \otimes V_2, \qquad \lambda u \otimes v = \lambda(u \otimes v) = u \otimes \lambda v \in V_1 \otimes V_2.$$

Elements of $V_1 \otimes V_2$ are of the form $\sum_{i=1}^{n} v_i^{(1)} \otimes v_i^{(2)}$, $v_i^{(1)} \in V_1$, $v_i^{(2)} \in V_2$, $1 \le i \le n$.
Elements of the form $v_1 \otimes v_2$ are called *pure tensors*.

Proposition (Tensor product of maps) Let $f : V \to V'$, $g : W \to W'$ be linear maps. There is a unique linear map $f \otimes g : V \otimes W \to V' \otimes W'$, $v \otimes w \longmapsto f(v) \otimes g(w)$.

Proof. The map $(v, w) \mapsto f(v) \otimes g(w)$ is bilinear.

Properties of tensor products.

Remark. In order to define linear maps $f : V_1 \otimes V_2 \longrightarrow W$, it is useful to just find a bilinear map $f : V_1 \times V_2 \longrightarrow W$ and then define $F : V_1 \otimes V_2 \longrightarrow W$ by universality.

Proposition Let $\{e_i\}_{i \in I}$ be a basis of V and $\{f_j\}_{j \in J}$ be a basis of W. Then $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is a basis of $V \otimes W$.

Proof. Since $\otimes$ is bilinear, one can easily see that $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ generates $V \otimes W$. Now assume that $\sum a_{ij} e_i \otimes f_j = 0$. Fix $i_0 \in I$, $j_0 \in J$. Now consider the map

$$f : V \times W \longrightarrow \mathbb{K}, \quad (v, w) \longmapsto e_{i_0}^*(v) f_{j_0}^*(w),$$

where $e_{i_0}^* : V \longrightarrow \mathbb{K}$, $e_i \longmapsto \begin{cases} 1 & i = i_0 \\ 0 & i \neq i_0 \end{cases}$ and analogously $f_{j_0}^* : W \longrightarrow \mathbb{K}$, $f_j \longmapsto \begin{cases} 1 & j = j_0 \\ 0 & j \neq j_0 \end{cases}$.

f is bilinear, so then there exists $F : V \otimes W \to \mathbb{K}$ linear such that $F(v \otimes w) = e_{i_0}^*(v) f_{j_0}^*(w)$.

Hence $0 = F(0) = F(\sum a_{ij} e_i \otimes f_j) = \sum a_{ij} e_{i_0}^*(e_i) f_{j_0}^*(f_j) = a_{i_0 j_0}$.

Hence $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is a linearly independent set. ∎

Proposition (Associativity). Let V_1, V_2, V_3 be vector spaces. The following map is an isomorphism of vector spaces:

$$(V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$
$$(v_1 \otimes v_2) \otimes v_3 \longmapsto v_1 \otimes (v_2 \otimes v_3)$$

. The two vector spaces will be identified and we will write $V_1 \otimes V_2 \otimes V_3$.

Proof. Fix $v_3 \in V_3$. Define $f_{v_3} : V_1 \times V_2 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$ such that $f_{v_3}(v_1, v_2) = v_1 \otimes (v_2 \otimes v_3)$. It is easy to see that it is bilinear. By universal property, there is a unique linear map $F_{v_3} : V_1 \otimes V_2 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$ such that $F_{v_3}(v_1 \otimes v_2) = v_1 \otimes (v_2 \otimes v_3)$. Thus the map

$$f : (V_1 \otimes V_2) \times V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$
$$(v_1 \otimes v_2, v_3) \longmapsto F_{v_3}(v_1 \otimes v_2)$$

is well-defined by extending by linearity on the first argument. Also f is bilinear, so that there is a unique linear map $F : (V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$ s.t. $(v_1 \otimes v_2) \otimes v_3 \longmapsto v_1 \otimes (v_2 \otimes v_3)$. Similarly we can construct a linear map the other way around, inverse to F, hence the isomorphism. $\square$

By induction, we have:

Proposition. Let $f : V_1 \times \cdots \times V_n \longrightarrow W$ be a n-multilinear map. Then there exists a unique linear map

$$V_1 \otimes \cdots \otimes V_n \longrightarrow W$$
$$v_1 \otimes \cdots \otimes v_n \longmapsto f(v_1, \dots, v_n).$$

We have then a bijection between the set of n-multilinear maps from $V_1 \times \cdots \times V_n \to W$ and the set of linear maps from $V_1 \otimes \cdots \otimes V_n \longrightarrow W$.

Proposition. Let V be a vector space. The following maps are isomorphisms:

$$\mathbb{K} \otimes V \longrightarrow V, \qquad V \otimes \mathbb{K} \longrightarrow V$$
$$\lambda \otimes v \longmapsto \lambda v, \qquad v \otimes \lambda \longmapsto \lambda v.$$

Remark. Let V and W be vector spaces, and V' and W' vector subspaces of V and W, resp. The linear map

$$V' \otimes W' \longrightarrow V \otimes W$$
$$v \otimes w \longmapsto v \otimes w$$

is injective. Then we can consider $V' \otimes W'$ as a subspace of $V \otimes W$.

Proposition Let U, V vector spaces, U_1, U_2 vector subspaces of U, V_1, V_2 vector subspaces of V.

i) $(U_1 + U_2) \otimes (V_1 + V_2) = U_1 \otimes V_1 + U_2 \otimes V_1 + U_1 \otimes V_2 + U_2 \otimes V_2$.

ii) $(U_1 \otimes V_1) \cap (U_2 \otimes V_2) = (U_1 \cap U_2) \otimes (V_1 \cap V_2)$.

iii) If $U = U_1 \oplus U_2$, then $U \otimes V = (U_1 \otimes V) \oplus (U_2 \otimes V)$

iv) If $V = V_1 \oplus V_2$ then $U \otimes V = (U \otimes V_1) \oplus (U \otimes V_2)$

Proof. i) Both subspaces of $U \otimes V$ are generated by tensors $u \otimes v$, with $u \in U_1 \cup U_2$, $v \in V_1 \cup V_2$. Thus they are both equal.

ii) Clearly $(U_1 \cap U_2) \otimes (V_1 \cap V_2) \subseteq (U_1 \otimes V_1) \cap (U_2 \otimes V_2)$.

Now, consider $\{e_i\}_{i \in I'}$ basis of $U_1 \cap U_2$, and complete it to obtain basis $\{e_i\}_{i \in I_1}$ of U_1, $\{e_i\}_{i \in I_2}$ of U_2, $I' = I_1 \cap I_2$, and complete $\{e_i\}_{i \in I' \cup (I_1 \setminus I') \cup (I_2 \setminus I')}$ to a basis of U. Analogously, $\{f_j\}_{j \in J}$ basis of V (J', J_1, J_2).

Let $x \in (U_1 \otimes V_1) \cap (U_2 \otimes V_2)$, and write $x = \sum_{\substack{i \in I \\ j \in J}} a_{i,j} \, e_i \otimes f_j$.

Let $i_0 \notin I'$. Assume $i_0 \notin I_1$. Then $e_{i_0}^*(U_1) = (0)$. Since $x \in U_1 \otimes V_1$, we have $(e_{i_0}^* \otimes \mathrm{id}_V)(x) = 0$. Then $0 = \sum_{i \in I, j \in J} a_{i,j} \, e_{i_0}^*(e_i) f_j = \sum_{j \in J} a_{i_0,j} f_j$.

Since $\{f_j\}_{j \in J}$ is a basis of V, then $a_{i_0,j} = 0 \quad \forall j \in J$. Analogously, if $j_0 \notin J'$, then $a_{i,j_0} = 0 \quad \forall i \in I$. Therefore $x = \sum_{\substack{i \in I' \\ j \in J'}} a_{i,j} \, e_i \otimes f_j \in (U_1 \cap U_2) \otimes (V_1 \cap V_2)$.

iii) From i), $U \otimes V = (U_1 \otimes V) + (U_2 \otimes V)$. From ii), $(U_1 \otimes V) \cap (U_2 \otimes V) = (U_1 \cap U_2) \otimes V = (0) \otimes V = (0)$.

iv) Similarly to iii). $\blacksquare$

Definition For any vector space V, the dual is defined by $V^* := \mathrm{Hom}(V,\mathbb{K}) := \{ f : V \to \mathbb{K} : f \text{ is linear} \}$.
For any $f : V \to W$ linear, the transpose $f^* : W^* \to V^*$ is defined by $f^*(\alpha) := \alpha \circ f$.
V^* is a vector space: $(f+g)(x) := f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$, $\forall x \in V$, $f, g \in V^*$, $\lambda \in \mathbb{K}$.

Proposition Let V, W be vector spaces. The following linear map is injective

$$\Theta : V^* \otimes W^* \longrightarrow (V \otimes W)^*$$
$$f \otimes g \longmapsto \left\{ \begin{array}{l} V \otimes W \longrightarrow \mathbb{K} \\ v \otimes w \longmapsto f(v)g(w) \end{array} \right.$$

The map Θ is bijective if V or W are finite-dimensional.

Proof. We see that Θ is well-defined. Let $(f,g) \in V^* \times W^*$. Consider the map $\begin{array}{l} V \times W \longrightarrow \mathbb{K} \\ (v,w) \longmapsto f(v)g(w) \end{array}$.
It is bilinear. By universal property, there is a unique linear map $\Theta'(f,g) : \begin{array}{l} V \otimes W \longrightarrow \mathbb{K} \\ v \otimes w \longmapsto f(v)g(w) \end{array}$.
Then we have a bilinear map $\Theta' : V^* \times W^* \longrightarrow (V \otimes W)^*$, and again by universal property, $\Theta : V^* \otimes W^* \to (V \otimes W)^*$ in the statement exists.

We prove Θ is injective. Take $F \in V^* \otimes W^*$ non-zero s.t. $\Theta(F) = 0$. Write $F = \sum_{i,j} a_{i,j} f_i \otimes g_j$, where $\{f_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ are lin. indep. sets. We know that for every $v \otimes w \in V \otimes W$, $0 = F(v \otimes w) = \sum_{i,j} a_{i,j} f_i(v) g_j(w)$. Fix $w \in W$. Then for every $v \in V$, $\sum_{i \in I} \left(\sum_{j \in J} a_{i,j} g_j(w) \right) f_i(v) = 0$. Since $\{f_i\}_{i \in I}$ are indep., then

$\sum_{j \in J} a_{i,j} g_j(w) = 0 \quad \forall i \in I$. Since $\{g_j\}_{j \in J}$ are indep, we have $a_{i,j} = 0$, $\forall i \in I, j \in J$,

then $F = 0$, so that Θ is injective.

Now, assume V is finite-dimensional. Take $\{e_i\}_{i \in I}$ basis of V and $\{f_j\}_{j \in J}$ basis of W. Also, $F \in (V \otimes W)^*$. For any $i \in I$, let $g_i : W \to \mathbb{K}$ linear map such that $g_i(f_j) = F(e_i \otimes f_j)$, $\forall j \in J$. Since I is finite, $\sum_{i \in I} e_i^* \otimes g_i \in V^* \otimes W^*$.
Also $\Theta\left(\sum_{i \in I} e_i^* \otimes g_i \right)(e_k \otimes f_\ell) = \sum_{i \in I} e_i^*(e_k) g_i(f_\ell) = g_k(f_\ell) = F(e_k \otimes f_\ell)$.

Since $\{e_k \otimes f_\ell\}_{k \in I, \ell \in J}$ is a basis of $V \otimes W$, we have that $F \in \mathrm{Im}(\Theta)$.
The proof is similar in the case that W is finite-dimensional. ■

Example. i) $\mathbb{R}^2 \otimes \mathbb{R}^2 \cong M_2(\mathbb{R})$ as vector spaces, with $M_n(\mathbb{R}) = \{ 2 \times 2 \text{ matrices with entries in } \mathbb{R} \}$
ii) $\mathbb{C} \otimes \mathbb{R}[x] \cong \mathbb{C}[x]$ as $\mathbb{R}$-vector spaces, with $\mathbb{K}[x] = \{ a_0 + a_1 x + \dots + a_n x^n : n \geq 0, \ a_i \in \mathbb{K} \}$
Indeed, consider the bilinear map $\begin{array}{l} \mathbb{C} \times \mathbb{R}[x] \longrightarrow \mathbb{C}[x] \\ (\lambda, p(x)) \longmapsto \lambda p(x) \end{array}$. Then there exists
$F : \mathbb{C} \otimes \mathbb{R}[x] \longrightarrow \mathbb{C}[x]$ $\mathbb{R}$-linear map. It is surjective and injective:
surjective: Any λx^n comes from (λ, x^n). By linearity, we obtain all $\mathbb{C}[x]$.
injective: Take $\sum_k \lambda_k \otimes p_k(x)$ s.t. $F(\sum_k \lambda_k \otimes p_k) = 0$. Write $\lambda_k = a_k + i b_k$.

By def. of F, we obtain $\sum_k (a_k + i b_k) p_k(x) = 0 \Rightarrow \sum_k a_k p_k = 0$ and $\sum_k b_k p_k(x) = 0$.

Hence $\sum_k \lambda_k \otimes p_k(x) = \sum_k (a_k + i b_k) \otimes p_k(x) = 1 \otimes \sum_k a_k p_k(x) + i \otimes \sum_k b_k p_k(x) = 0$. Thus F is injective. ■

2. Algebras and Coalgebras

Intuitively, an algebra A is a vector space together with an associative product $m : A \times A \to A$ compatible with the vector space operations. This conditions imply that m is bilinear. Then we can find a linear map $m : A \otimes A \to A$ st. $m(a \otimes b) = a \cdot b$.
The associativity of m can be written as follows: $a(bc) = (ab)c$

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id \otimes m"] \arrow[d, "m \otimes id"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

$m \circ (m \otimes id) = m \circ (id \otimes m)$

For the axioms for the unit, consider $\eta : \mathbb{k} \longrightarrow A$ linear. It is injective $(A \neq (0))$ and we can identify $\mathbb{k}$ as a subalgebra. $\lambda \longmapsto \lambda 1_A$

identifications given in a previous thm.

$$\begin{tikzcd} \mathbb{k}\otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "id"'] & A\otimes A \arrow[d, "m"] & A\otimes \mathbb{k} \arrow[l, "id\otimes\eta"'] \arrow[dl, "id"] \\ & A & \end{tikzcd}$$

Then we can write $a \cdot 1_A = a = 1_A \cdot a$

$$m\circ(\eta\otimes id) = id = m\circ(id\otimes\eta).$$

Definition. An <u>algebra</u> is a triple (A, m, η) where A is a vector space, $m: A\otimes A \longrightarrow A$ is a linear map called <u>multiplication</u>, $\eta: \mathbb{k}\to A$ is a linear map called <u>unit</u>, that satisfy the following conditions:

- Associativity $\quad m\circ(id\otimes m) = m\circ(m\otimes id)$

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

- Unity $\quad m\circ(id\otimes\eta) = id = m\circ(\eta\otimes id)$

$$\begin{tikzcd} \mathbb{k}\otimes A \cong A \cong A\otimes\mathbb{k} \arrow[r, "id\otimes\eta"] \arrow[d, "\eta\otimes id"'] \arrow[dr, "id"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"] & A \end{tikzcd}$$

Proposition. Let (A, m, η) be an algebra.

i) A subalgebra of A is a subspace B of A such that $m(B\otimes B)\subseteq B$ and $\eta(\mathbb{k})\subseteq B$.

ii) An (bilateral) ideal of A is a subspace I of A s.t. $m(A\otimes I + I\otimes A)\subseteq I$.

Proposition. Let A, B be algebras and $f: A\to B$ a linear map. Then f is an <u>algebra morphism</u> if and only if

i) $m_B\circ(f\otimes f) = f\circ m_A$

$$\begin{tikzcd} A\otimes A \arrow[r, "f\otimes f"] \arrow[d, "m_A"'] & B\otimes B \arrow[d, "m_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

ii) $f\circ\eta_A = \eta_B$

$$\begin{tikzcd} \mathbb{k} \arrow[r, "\eta_A"] \arrow[dr, "\eta_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

Proposition. An algebra A is <u>commutative</u> if and only if $m\circ\tau = m$, where $\tau: A\otimes A \longrightarrow A\otimes A$, $a\otimes b \longmapsto b\otimes a$ is the <u>flip</u>.

Example. Let V be a vector space. For any $n\geq 1$, we write $V^{\otimes n} := \underbrace{V\otimes\cdots\otimes V}_{n \text{ times}}$. By convention $V^{\otimes 0} := \mathbb{k}$. An element in $V^{\otimes n}$ is a linear combination of tensors of length n. Such tensors are called <u>words</u> in the alphabet V of lenght n.

Definition. Let V be a vector space. The <u>tensor algebra</u> of V is $T(V) := \bigoplus_{n\geq 0} V^{\otimes n}$.

To simplify notation, we write $v_1\cdots v_n$ instead of $v_1\otimes\cdots\otimes v_n$.

Proposition. Let $\{v_i\}_{i\in I}$ be a basis of V. A basis of $T(V)$ is given by the words on the alphabet $\{v_i\}_{i\in I}$: $\quad \{v_{i_1} v_{i_2}\cdots v_{i_k}\}_{\substack{k\geq 0 \\ i_1,\dots,i_k\in I}}$, where if $k=0$, we obtain the empty word 1.

Theorem. Let V be a vector space. $T(V)$ is an algebra with product given by concatenation of words: $(v_1\cdots v_k, w_1\cdots w_\ell) \longmapsto v_1\cdots v_k w_1\cdots w_\ell$. The empty word is the unit for the concatenation product. Also, $T(V)$ satisfies the following universal property: if A is an algebra and $f: V\to A$ is a linear map, then there is a unique algebra morphism $F: T(V)\longrightarrow A$ such that $F\circ i = f$, where $i: V\longrightarrow T(V)$ is the natural inclusion.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & A \\ T(V) \arrow[ur, dashed, "\exists ! F"'] & \end{tikzcd}$$

Proof. We will prove the universal property. The map $(v_1, \dots, v_n) \longmapsto f(v_1)\cdots f(v_n)$ is n-multilinear. Then there is a unique linear map $F_n : V^{\otimes n} \longrightarrow A$, $v_1 \cdots v_n \longmapsto f(v_1)\cdots f(v_n)$. Thus, we obtain a map

$F : T(V) \longrightarrow A$, $v_1 \cdots v_n \longmapsto F(v_1)\cdots F(v_n)$. It is clear that F is an algebra morphism such that $f(v) = F(v)$. Then,

if F' another algebra morphism satisfying this property, we have:

$$F'(v_1 \cdots v_n) = F'(v_1)\cdots F'(v_n) = f(v_1)\cdots f(v_n) = F(v_1)\cdots F(v_n) = F(v_1 \cdots v_n). \Rightarrow F = F'. \quad \blacksquare$$

Remark. If X is a set, then we have $\Bbbk\langle X\rangle = T(\Bbbk X)$ where $\Bbbk\langle X\rangle$ is the non-commutative polynomial algebra on indeterminates X.

To define the notion of coalgebra, we dualize the axioms in the definition of algebra.

Definition A <u>coalgebra</u> (counital coassociative) is a triple (C, Δ, ε) where C is a vector space, $\Delta : C \longrightarrow C \otimes C$ is a linear map called <u>comultiplication</u> and $\varepsilon : C \longrightarrow \Bbbk$ is a linear map called <u>counit</u> such that the following conditions hold:

i) Coassociativity $(\Delta \otimes id)\circ \Delta = (id \otimes \Delta)\circ \Delta$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes id"] \\ C \otimes C \arrow[r, "\Delta \otimes id"] & C \otimes C \otimes C \end{tikzcd}$$

ii) Counity $(\varepsilon \otimes id)\circ \Delta = id = (id \otimes \varepsilon)\circ \Delta$

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[drr, "id"] & C \otimes C \arrow[dr, "id \otimes \varepsilon"] & \\ C \otimes C \arrow[r, "\varepsilon \otimes id"] & \Bbbk \otimes C & \cong C \cong C \otimes \Bbbk \end{tikzcd}$$

If $\tau \circ \Delta = \Delta$, we say that C is <u>cocommutative</u>.

Example i) Let X be a set. Then $(\Bbbk S, \Delta, \varepsilon)$ is a coalgebra, where $\Delta(s) = s \otimes s$ and $\varepsilon(s) = 1$. <u>Check.</u> It is cocommutative.

ii) (Incidence coalgebra). Let P be a poset. For $x \le y \in P$, define the interval $[x,y] := \{z \in P : x \le z \le y\}$. Set $\mathrm{Int}(P) = \{\text{intervals in } P\} = \{[x,y] : x \le y \text{ in } P\}$. Then $C := \Bbbk\, \mathrm{Int}(P)$.

Finally, $\Delta([x,y]) = \sum_{\substack{z \in P \\ z \in [x,y]}} [x,z] \otimes [z,y]$, $\quad \varepsilon([x,y]) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$, extend linearly,

(C, Δ, ε) is a coalgebra called the incidence coalgebra.

We can check:

$$(\Delta \otimes id)\Big(\sum_{x \le z \le y} [x,z] \otimes [z,y]\Big) = \sum_{x \le z \le y} \Big(\sum_{x \le z' \le z} [x,z'] \otimes [z',z]\Big) \otimes [z,y] = \sum_{x \le z' \le z \le y} [x,z'] \otimes [z',z] \otimes [z,y]$$

$$(id \otimes \Delta)\Big(\sum_{x \le z \le y} [x,z] \otimes [z,y]\Big) = \cdots = \sum_{x \le z \le z' \le y} [x,z] \otimes [z,z'] \otimes [z',y] \quad //$$

iii) (Matrices) For $n \ge 1$, write $M_n^{\vee}(\Bbbk)$ the vector space with basis $\{e_{i,j}\}_{1 \le i,j \le n}$.

Define $\Delta(e_{i,j}) = \sum_{k=1}^{n} e_{i,k} \otimes e_{k,j}$. It is coassociative:

$$(\Delta \otimes id)\circ \Delta(e_{i,j}) = \sum_{k=1}^{n} \Delta(e_{i,k}) \otimes e_{k,j} = \sum_{k,l=1}^{n} e_{i,l} \otimes e_{l,k} \otimes e_{k,j} = \sum_{l=1}^{n} e_{i,l} \otimes \Delta(e_{l,j}) = (id \otimes \Delta)\circ \Delta(e_{i,j}).$$

If $\varepsilon(e_{i,j}) = \delta_{i,j}$, then $(\varepsilon \otimes id)\circ \Delta(e_{i,j}) = \sum_{k=1}^{n} \delta_{i,k} e_{k,j} = e_{i,j} = (id \otimes \varepsilon)\circ \Delta(e_{i,j})$.

Hence $M_n^{\vee}(\Bbbk)$ is a coalgebra. For $n \ge 2$, it is non-cocommutative.

Sweedler notation: Let C be a coalgebra. Since $\Delta: C \to C\otimes C$, then we have $\Delta(c) = \sum_{i=1}^{n} c_{1,i}\otimes c_{2,i}$, $\forall c\in C$. We will write $\Delta(c) = \sum_{(c)} c_{(1)}\otimes c_{(2)}$. Coassociativity writes

$$\sum_{(c)}\sum_{(c_1)} (c_{(1)})_{(1)}\otimes (c_{(1)})_{(2)}\otimes c_{(2)} = \sum_{(c)}\sum_{(c_2)} c_{(1)}\otimes (c_{(2)})_{(1)}\otimes (c_{(2)})_{(2)}$$

$$=: \sum_{(c)} c_{(1)}\otimes c_{(2)}\otimes c_{(3)} =: \Delta^{[2]}(c).$$

More generally, the iterated coproduct can be written: $(\Delta^{[1]} := \Delta)$

$$\Delta^{[n]}(c) := (\Delta\otimes id^{\otimes(n-1)})\circ\Delta^{[n-1]}(c) = \sum_{(c)} c_{(1)}\otimes c_{(2)}\otimes\cdots\otimes c_{(n+1)}, \quad n\geq 2.$$

Coassociativity says to us that it does not matter in which component of the tensor we iterate Δ.

Counit property writes: $\sum_{(c)}\varepsilon(c_{(1)})c_{(2)} = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}) = c.$

Algebras and coalgebras are more o less equivalent objects.

Proposition i) Let (C,Δ,ε) be a coalgebra. Then C^* is an algebra with multiplication $(fg)(x) = (f\otimes g)\circ\Delta(x) = \sum_{(x)} f(x_{(1)})g(x_{(2)})$. The unit is given by ε. [$C^*\otimes C^*\subseteq (C\otimes C)^*$]

ii) Let (A,m,η) be a finite-dimensional algebra. Then A^* is a coalgebra with $\Delta = m^*: A^*\longrightarrow (A\otimes A)^* = A^*\otimes A^*$. The counit is given by $\varepsilon(f) := f(1)$.

Proof. Associativity of the product follows from coassociativity of Δ. Indeed, in Sweedler notation $f,g,h\in C^*$

$$((f\cdot g)h)(x) = \sum_{(x)}(fg)(x_{(1)})h(x_{(2)}) = \sum_{(x)}\sum_{(x_{(1)})} f((x_{(1)})_{(1)})\,g((x_{(1)})_{(2)})\,h(x_{(2)})$$

$$= \sum_{(x)}\sum_{(x_2)} f(x_{(1)})\,g((x_{(2)})_{(1)})\,h((x_{(2)})_{(2)}) = \sum_{(x)} f(x_{(1)})(gh)(x_{(2)})$$

$$= f(gh)(x).$$

$$(f\cdot g)h = (f\cdot g\otimes h)\circ\Delta = ((f\otimes g)\circ\Delta\otimes h)\circ\Delta$$
$$= (f\otimes g\otimes h)\circ(\Delta\otimes id)\circ\Delta$$
$$= (f\otimes g\otimes h)\circ(id\otimes\Delta)\circ\Delta \quad \text{by coassociativity}$$
$$= (f\otimes gh)\circ\Delta = f\cdot(gh).$$

For the unit $\varepsilon\cdot f = (\varepsilon\otimes f)\circ\Delta = (id_{\mathbb{K}}\otimes f)\circ(\varepsilon\otimes id)\circ\Delta = f\circ id = f = (f\otimes\varepsilon)\circ\Delta = f\cdot\varepsilon.$

ii) Since A is finite-dimensional, we can identify $(A\otimes A\otimes A)^*\cong A^*\otimes A^*\otimes A^*$. Then if $x\otimes y\otimes z\in A\otimes A\otimes A$ and $f\in A^*$:

$\Delta = m^*: A^*\longrightarrow (A\otimes A)^* = A^*\otimes A^*$

$\Delta(f) = m^*(f) = f\circ m$ by def.

$(\Delta\otimes id)\circ\Delta = (m^*\otimes id^*)\circ m^* = (m\circ(m\otimes id))^*$

$$(\Delta\otimes id)\circ\Delta(f)\,(x\otimes y\otimes z) = \Delta(f)(xy\otimes z)$$
$$= f((xy)\cdot z)$$
$$= f(x\cdot(yz))$$
$$= \Delta(f)(x\otimes yz)$$
$$= (id\otimes\Delta)\circ\Delta(f)\,(x\otimes y\otimes z).$$

transpose of compositions $(g\circ h)^* = h^*\circ g^*$

$(g\circ h)^*(\alpha) := \alpha\circ(g\circ h)$
$= (\alpha\circ g)\circ h$
$= g^*(\alpha)\circ h$
$= (h^*\circ g^*)(\alpha)$

For the unit $\varepsilon(f) = f(1)$, we have

$(\varepsilon\otimes id)\circ\Delta(f)(x) \overset{A\cong\mathbb{K}\otimes A}{=} \Delta(f)(1\otimes x) = f(1\cdot x) = f(x)$

Analogously, $(id\otimes\varepsilon)\circ\Delta(f)(x) = f(x).$

$\varepsilon = \eta^*: A^*\longrightarrow\mathbb{K}$

$\alpha\longmapsto\alpha\circ\eta$

$\lambda\overset{\eta}{\longmapsto}\lambda 1_A\overset{\alpha}{\longmapsto}\lambda\alpha(1)$

__Example.__ The dual of the coalgebra $M_n^*(\mathbb{K})$ is the algebra of $n\times n$ matrices $M_n(\mathbb{C})$.

Actually, if $\{E_{i,j}\}_{1\leq i,j\leq n}$ is the dual basis of $\{e_{i,j}\}_{1\leq i,j\leq n}$, and if we write $E_{i,j}E_{k,l} = \sum_{1\leq s,t\leq n} a_{s,t}E_{s,t}$, then $a_{s,t} = E_{i,j}E_{k,l}(e_{s,t}) = (E_{i,j}\otimes E_{k,l})\circ\Delta(e_{s,t}) = \sum_{u=1}^{n}\delta_{i,s}\delta_{j,u}\delta_{k,u}\delta_{l,t}$

Then $E_{i,j}E_{k,l} = 0$ if $j\neq k$ and $E_{i,j}E_{k,l} = E_{i,l}$ if $j=k$.

Example (Incidence algebra of a poset P). Let $C(P)$ be the incidence coalgebra of P. We will call its dual $A(P) := (C(P))^*$ the <u>incidence algebra</u> of P.

Elements: linear functionals $c^*: C \longrightarrow \mathbb{k} \iff$ functions $c^*: \mathrm{Int}(P) \longrightarrow \mathbb{k}$.

Multiplication: $c^* d^* ([x,y]) = c^* \otimes d^* \circ \Delta([x,y]) = c^* \otimes d^* \left(\sum_{x \le z \le y} [x,z] \otimes [z,y] \right) = \sum_{x \le z \le y} c^*([x,z])\, d^*([z,y])$.

Unit: $\varepsilon([x,y]) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} =: \delta([x,y])$.

Definition Let C be a coalgebra and V be a vector subspace of C.

i) V is a <u>sub-coalgebra</u> of V if $\Delta(V) \subseteq V \otimes V$.

ii) V is a <u>two-sided coideal</u> if $\Delta(V) \subseteq V \otimes C + C \otimes V$ and $\varepsilon(V) = (0)$.

Proposition Let C be a coalgebra and V be a subspace of C.

i) If V is a subcoalgebra of V, then $(V, \Delta|_V, \varepsilon|_V)$ is a coalgebra.

ii) If V is a coideal, then C/V has a coalgebra structure given by $\Delta'(\bar{x}) := \sum_{(x)} \bar{x}_{(1)} \otimes \bar{x}_{(2)}$, $\varepsilon'(\bar{x}) = \varepsilon(x)$.

Proof. ii) We will show that Δ' and ε' are well-defined. Then the coalgebra axioms of C will imply the coalgebra axioms of C/V.

Since $\varepsilon(V) = (0)$, then $V \subseteq \ker(\varepsilon)$, so that $\varepsilon: C/V \longrightarrow \mathbb{k}$ is well-defined. Now, take $x \in V$. Then $\Delta(x) \in V \otimes C + C \otimes V$. Then we can write: $\Delta(x) = \underbrace{\sum_{(x)} x_{(1)} \otimes x_{(2)}}_{\in V \otimes C} + \underbrace{\sum_{(y)} y_{(1)} \otimes y_{(2)}}_{\in C \otimes V}$

Then $\sum_{(x)} \underbrace{\overline{x_{(1)}}}_{=\bar{0}} \otimes \overline{x_{(2)}} + \sum_{(y)} \overline{y_{(1)}} \otimes \underbrace{\overline{y_{(2)}}}_{=0} = 0 = \Delta'(\bar{x})$. Hence Δ' is well-defined. ■

Definition Let C, D be two coalgebras and $f: C \longrightarrow D$ be a linear map. We say that f is a <u>coalgebra morphism</u> if $\Delta_D \circ f = (f \otimes f) \circ \Delta_C$ and $\varepsilon_D \circ f = \varepsilon_C$

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\varepsilon_C"] \arrow[d, "f"'] & \mathbb{k} \\ D \arrow[ur, "\varepsilon_D"'] & \end{tikzcd}$$

Proposition If $f: C \longrightarrow D$ is a coalgebra morphism then $f^*: D^* \longrightarrow C^*$ is an algebra morphism.

Proof. Let $\alpha, \beta \in D^*$. Then we have:

$f^*(\alpha\beta) = (\alpha \cdot \beta) \circ f$

def. of $\cdot$ $= (\alpha \otimes \beta) \circ (\Delta_D \circ f)$

since f is a coalg. morphism $= (\alpha \otimes \beta) \circ (f \otimes f) \circ \Delta_C$ $\quad = \sum_{(c)} (\alpha \circ f)(c_{(1)}) \cdot (\beta \circ f)(c_{(2)})$ maybe more intuitive using Sweedler notation.

$= (\alpha \circ f) \otimes (\beta \circ f) \circ \Delta_C \quad = (f^*(\alpha) \otimes f^*(\beta)) \circ \Delta_C \quad = f^*(\alpha) \cdot f^*(\beta)$

Proposition If $f: A \to B$ is an algebra morphism with A, B finite-dimensional vector spaces, then $f^*: B^* \longrightarrow A^*$ is a coalgebra morphism.

•) Goal: prove a version of the first isomorphism theorem for coalgebras.

Lemma Let $f: V \longrightarrow V'$, $g: W \longrightarrow W'$ be linear maps and consider $f \otimes g: V \otimes W \longrightarrow V' \otimes W'$. Then

i) $\mathrm{Im}(f \otimes g) = \mathrm{Im}(f) \otimes \mathrm{Im}(g)$. $\qquad$ ii) $\ker(f \otimes g) = \ker f \otimes W + V \otimes \ker g$.

Proof. Exercise :) . ■

Proposition: Let $f: C \to D$ be a coalgebra morphism. Then $\ker(f)$ is a coideal of C and $\mathrm{Im}(f)$ is a subcoalgebra of D.

Proof. Let $c \in \ker(f)$. Then $f(c) = 0 \Rightarrow 0 = \Delta_D f(c) = (f \otimes f) \circ \Delta_C(c)$ since f is a coalgebra morphism. Then $\Delta_C(c) \in \ker(f \otimes f) = \ker f \otimes C + C \otimes \ker f$.

Now let $f(c) \in \mathrm{Im}(f)$. Then $\Delta_D \circ f(c) = (f \otimes f) \circ \Delta_C(c) = \sum_{(c)} f(c_{(1)}) \otimes f(c_{(2)}) \in \mathrm{Im}(f) \otimes \mathrm{Im}(f)$. ■

Proposition (Fundamental iso. thm. for coalgebras) If $f: C \longrightarrow D$ is a coalgebra morphism, then $\mathrm{Im}\, f \cong C/\ker f$ as coalgebras.

Proof. $\bar{f}: C/\ker f \longrightarrow \mathrm{Im}\, f$, $\bar{c} \longmapsto f(c)$ is a coalgebra morphism since the quotient coalg. structure of $C/\ker f$ is inherited from C. ■

The following theorem provides a fundamental property in the structure of coalgebras that contrasts with the structure of algebras.

Theorem (Fundamental Theorem of Coalgebras) Let C be a coalgebra and $x \in C$. Then there exists a subcoalgebra $D \subseteq C$ such that $x \in D$ and $\dim_{\mathbb{K}} D < \infty$.

Proof. Let $\Delta(x) = \sum_i b_i \otimes c_i$. We consider $\Delta_2(x) = \sum_i \Delta(b_i) \otimes c_i = \sum_{i,j} a_j \otimes b_{i,j} \otimes c_i$.
We may assume that $\{a_j\}_j$ are linearly independent and so are $\{c_i\}_{i\in I}$.
Let D be the subspace generated by $\{b_{i,j}\}_{i,j}$. We claim that

$x = \sum_{i,j} \varepsilon(a_j)\,\varepsilon(c_i)\, b_{i,j}$. Indeed, notice that

$$\begin{aligned}(\varepsilon \otimes id \otimes \varepsilon) \circ \Delta_2 &= (\varepsilon \otimes id \otimes \varepsilon) \circ (\Delta \otimes id) \circ \Delta \\ &= [((\varepsilon \otimes id)\circ \Delta) \otimes \varepsilon] \circ \Delta \overset{\text{def. of counit}}{=} (id \otimes \varepsilon)\circ \Delta \overset{\text{def. of counit}}{=} id\end{aligned}$$

Hence $x \in D$. We will show that D is a subcoalgebra, i.e. $\Delta(D) \subseteq D \otimes D$.
Indeed, by coassociativity, we have that $\sum_{i,j} \Delta(a_j) \otimes b_{i,j} \otimes c_i = \sum_{i,j} a_j \otimes \Delta(b_{i,j}) \otimes c_i$

Since $\{c_i\}_i$ are linearly independent, we obtain $\sum_j \Delta(a_j) \otimes b_{i,j} = \sum_j a_j \otimes \Delta(b_{i,j})$, $\forall i \in I$.
Then $\sum_j a_j \otimes \Delta(b_{i,j}) \in C \otimes C \otimes D$. Then by Exercise 3, List 1, we have that $\Delta(b_{i,j})$ is in $C \otimes D$. Analogously, we can show that $\Delta(b_{i,j}) \in D \otimes C$. Hence
$\Delta(b_{i,j}) \in C \otimes D \cap D \otimes C = D \otimes D$, and we conclude. ■

Remark. Ex 3 from list 1 follows from the fact that if U, V are vector subspaces, and $V' \subseteq V$ is a subspace, then $U \otimes V/_{V'} \cong (U \otimes V)/(U \otimes V')$. This can be proved by considering the map
$$\begin{aligned} U \otimes V &\longrightarrow U \otimes V/_{V'} \\ u \otimes v &\longmapsto id_U(u) \otimes \pi_{V'}(v) \end{aligned}$$
and showing that $\ker(id_U \otimes \pi_{V'}) = U \otimes V'$.
Then, we can use the fact that if $0 = \sum_{i=1}^{n} u_i \otimes \bar{v}_i \in U \otimes V/_{V'}$ and $\{u_i\}_i$ are linearly independent, then $\bar{v}_i = 0 \in V/_{V'} \Rightarrow v_i \in V'$. In particular, if $x = \sum_{i=1}^{n} u_i \otimes v_i \in U \otimes V' \subseteq U \otimes V$, then $\pi_{U\otimes V'} : U \otimes V \twoheadrightarrow U \otimes V / U \otimes V'$ maps $x \mapsto 0$, so that we can use the above argument.

Example. Let V be a vector space. The tensor algebra $T(V)$ has a coalgebra structure when equipped with the <u>deconcatenation coproduct</u> $\Delta_d(v_1 \cdots v_n) = \sum_{p=0}^{n} v_1 \cdots v_p \otimes v_{p+1} \cdots v_n$ and counit given by $\varepsilon(1) = 1$, $\varepsilon(v_1 \cdots v_n) = 0$ if $n \geq 1$. (we set $v_1 \cdots v_0 = v_{n+1} \cdots v_n = 1$).

We check: $(\Delta_d \otimes id) \circ \Delta_d(v_1 \cdots v_n) = \sum_{p=0}^{n} \Delta_d(v_1 \cdots v_p) \otimes v_{p+1} \cdots v_n = \sum_{p=0}^{n} \sum_{p_1=0}^{p} v_1 \cdots v_{p_1} \otimes v_{p_1} \cdots v_p \otimes v_{p+1} \cdots v_n$.

$= $ changing the order of the double sum.

$(id \otimes \Delta_d) \circ \Delta_d(v_1 \cdots v_n) = \sum_{p=0}^{n} v_1 \cdots v_p \otimes \Delta_d(v_{p+1} \cdots v_n) = \sum_{p=0}^{n} \sum_{p_1=p}^{n} v_1 \cdots v_p \otimes v_{p+1} \cdots v_{p_1} \otimes v_{p_1+1} \cdots v_n$

Example. The tensor algebra $T(V)$ has another coalgebra structure when equipped with the <u>unshuffle coproduct</u> $\Delta_u(v_1 \cdots v_n) = \sum_{I \subseteq [n]} v_I \otimes v_{[n]\setminus I}$, where if $I = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$, then $v_I := v_{i_1} \cdots v_{i_p}$. The counit is defined by $\varepsilon(1) = 1$ and $\varepsilon(v_1 \cdots v_n) = 0$ if $n \geq 1$.

Bialgebras

Definition A <u>bialgebra</u> is a tuple $(B, m, \eta, \Delta, \varepsilon)$ such that (B, m, η) is an algebra, (B, Δ, ε) is a coalgebra, such that the following diagrams commute:

$\tau(a \otimes b) = b \otimes a$

$$\begin{tikzcd} B \otimes B \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\ B \otimes B \otimes B \otimes B \arrow[rr, "id \otimes \tau \otimes id"] & & B \otimes B \otimes B \otimes B \arrow[u, "m \otimes m"'] \end{tikzcd}$$

$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$

$$\begin{tikzcd} B \otimes B \arrow[rr, "m"] \arrow[dr, "\varepsilon \otimes \varepsilon"'] & & B \arrow[dl, "\varepsilon"] \\ & \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} & \end{tikzcd}$$

$\varepsilon \circ m = \varepsilon \otimes \varepsilon$

$$\begin{tikzcd} \mathbb{K}\otimes\mathbb{K} \cong \mathbb{K} \arrow[d, "\eta\otimes\eta"'] \arrow[drr, "\eta"] & & \\ B\otimes B & & B \arrow[ll, "\Delta"'] \end{tikzcd} \qquad \eta\otimes\eta = \Delta\circ\eta$$

$$\begin{tikzcd} \mathbb{K} \arrow[dr, "\eta"] \arrow[dd, "id"'] & \\ & B \arrow[dl, "\varepsilon"] \\ \mathbb{K} & \end{tikzcd} \qquad id_{\mathbb{K}} = \varepsilon\circ\eta .$$

Remark. i) Let (A, m_A, η_A) and (B, m_B, η_B) be two algebras. The tensor product $A\otimes B$ has an algebra structure given by $m_{A\otimes B} := (m_A\otimes m_B)\circ(id_A\otimes\tau\otimes id_B)$, and $\eta_{A\otimes B} = \eta_A\otimes\eta_B : \mathbb{K}\otimes\mathbb{K}\cong\mathbb{K}\longrightarrow A\otimes B$. In particular, we can write $(a\otimes b)\cdot(a'\otimes b') = a\cdot a'\otimes b\cdot b'$, for any $a, a'\in A$, $b, b'\in B$.

ii) Analogously, let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be two coalgebras. The tensor product has a coalgebra structure given by $\Delta_{C\otimes D} := (id_C\otimes\tau\otimes id_C)\circ(\Delta_C\otimes\Delta_D)$ and $\varepsilon_{C\otimes D} := \varepsilon_C\otimes\varepsilon_D$.

The compatibility diagrams in the definition of a bialgebra describe the relation of the maps m and η with the coalgebra structure and also the relation of Δ and ε with the algebra structure.

Lemma. Let B be a vector space such that (B, m, η) is an algebra and (B, Δ, ε) is a coalgebra. The following are equivalent:

i) Δ and ε are algebra morphisms.

ii) m and η are coalgebra morphisms

iii) For any $x, y\in B$, $\Delta(xy) = \sum_{(x)}\sum_{(y)} x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)}$, $\Delta(1_B) = 1_B\otimes 1_B$, $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$, $\varepsilon(1_B) = 1_{\mathbb{K}}$

Proof. i) $\Leftrightarrow$ iii) $\Delta: B\to B\otimes B$ is an algebra morphism if and only if $\Delta(xy) = \Delta(x)\Delta(y) = \sum_{(x),(y)} x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)}$ and $\Delta(1) = 1_{B\otimes B} = 1\otimes 1$. In the same way, $\varepsilon: B\to\mathbb{K}$ is an algebra morphism if and only if $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ $\forall x, y\in B$ and $\varepsilon(1) = 1_{\mathbb{K}}$. Hence i) and iii) are equivalent.

ii) $\Leftrightarrow$ iii) $m: B\otimes B\to B$ is a coalgebra morphism if and only if for any $x\otimes y\in B\otimes B$:

$$\Delta\circ m(x\otimes y) = (m\otimes m)\circ\Delta_{B\otimes B}(x\otimes y)$$

$$\Delta(xy) = (m\otimes m)\circ\Big(\sum_{(x),(y)} x_{(1)}\otimes y_{(1)}\otimes x_{(2)}\otimes y_{(2)}\Big) = \sum_{(x),(y)} x_{(1)}y_{(1)}\otimes x_{(2)}y_{(2)} .$$

and for any $x\otimes y\in B\otimes B$, $\varepsilon\circ m(x\otimes y) = \varepsilon(xy) \overset{def}{=} \varepsilon_{B\otimes B}(x\otimes y) = \varepsilon(x)\,\varepsilon(y)$.

Also, $\eta:\mathbb{K}\to B$ is a coalgebra morphism if and only if:

$$\Delta\circ\eta\,(1_{\mathbb{K}}) = (\eta\otimes\eta)\circ\Delta_{\mathbb{K}}(1_{\mathbb{K}})$$

$$\Delta(1_B) = (\eta\otimes\eta)(1_{\mathbb{K}}\otimes 1_{\mathbb{K}}) = \eta(1_{\mathbb{K}})\otimes\eta(1_{\mathbb{K}}) = 1_B\otimes 1_B ,$$

recall that $(\mathbb{K}, \Delta_{\mathbb{K}}, \varepsilon_{\mathbb{K}})$ is a coalgebra with $\Delta_{\mathbb{K}}(1_{\mathbb{K}}) = 1\otimes 1$ and $\varepsilon = id_{\mathbb{K}}$.

and also $\varepsilon\circ\eta(1_{\mathbb{K}}) = \varepsilon(1_B) = \varepsilon_{\mathbb{K}}(1_{\mathbb{K}}) = 1_{\mathbb{K}}$. ∎

Proposition. A tuple $(B, m, \eta, \Delta, \varepsilon)$ is a bialgebra if and only if, (B, m, η) is an algebra, (B, Δ, ε) is a coalgebra, and Δ and ε are algebra morphisms.

Example Group bialgebra. Let G be a group and consider $B = \mathbb{K}G$. The product on G is linearly extended to a product $m: B\otimes B\longrightarrow B$. Set $\eta: \mathbb{K}\to B$, $1\mapsto e$, $\Delta(g) = g\otimes g$ $\forall g\in G$ and extend linearly. Finally, set $\varepsilon: B\longrightarrow\mathbb{K}$, $g\longmapsto 1$

We have that B is a bialgebra:

$$\begin{tikzcd} g\otimes h \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & gh \arrow[r, "\Delta"] & gh\otimes gh \\ g\otimes g\otimes h\otimes h \arrow[rr, "id\otimes\tau\otimes id"] & & g\otimes h\otimes g\otimes h \arrow[u, "m\otimes m"'] \end{tikzcd}$$

Check the other three diagrams.

Example: (Polynomial ring) Consider $B = \mathbb{K}[x]$. multiplication and unit usual, $\varepsilon(x^n) = \begin{cases} 1 & n=0 \\ 0 & n \geq 1 \end{cases}$

Coproduct $\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$. Comes from extending $\Delta(x) = 1 \otimes x + x \otimes 1$ multiplicatively.

Hence B is a bialgebra. Thus we have

$$\begin{tikzcd} x^a \otimes x^b \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"] & x^{a+b} \arrow[r] & \sum_{k=0}^{a+b} \binom{a+b}{k} x^k \otimes x^{a+b-k} \\ \left(\sum_{i=0}^{a} \binom{a}{i} x^i \otimes x^{a-i}\right) \otimes \left(\sum_{j=0}^{b} \binom{b}{j} x^j \otimes x^{b-j}\right) \arrow[rr, "id\otimes\tau\otimes id"] & & \sum \binom{a}{i}\binom{b}{j} x^i \otimes x^j \otimes x^{a-i} \otimes x^{b-j} \arrow[u, "m\otimes m"'] \end{tikzcd}$$

By comparing coefficients, we have that $\binom{a+b}{k} = \sum_{i+j=k} \binom{a}{i}\binom{b}{j}$.

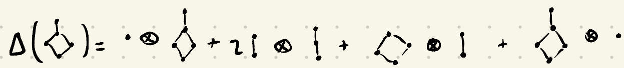
Example (Posets) Let $I = \mathbb{K}\{$ isomorphism classes of posets with $\hat{0}$ (minimum) and $\hat{1}$ (maximum element)$\}$.

Coalgebra: $\Delta(P) = \sum_{p \in P} [\hat{0}, p] \otimes [p, \hat{1}]$, $\varepsilon(P) = \begin{cases} 1 & \text{if } \bar{P} = \cdot \text{ (poset with one element)} \\ 0 & \text{otherwise} \end{cases}$

we have written $\bar{P}$ as the isomorphism class of the poset P.

Algebras: $m(P \otimes Q) = P \times Q =: P \cdot Q$, where $P \times Q$ stands for the direct product of posets:
$P \times Q = \{(p,q) : p \in P, q \in Q\}$, $(p,q) \leq (p',q') \Leftrightarrow p \leq p'$ and $q \leq q'$.

We have for instance:



•) Δ is an algebra morphism:

$$\Delta(P \times Q) = \sum_{(p,q) \in P\times Q} [(0,0),(p,q)] \otimes [(p,q),(1,1)] = \sum_{(p,q) \in P \times Q} ([0,p] \times [0,q]) \otimes ([p,1] \times [q,1])$$

multiplication in $A \otimes B$

$$= \sum_{p \in P, q \in Q} ([0,p] \otimes [p,1]) \times ([0,q] \otimes [q,1]) = \Delta(P) \times \Delta(Q).$$

Remark. For our last three examples, we have

	comm	cocomm
1	no	yes
2	yes	yes
3	yes	no

Remark Let B be a bialgebra of finite dimension. Then the dual $(B^*, \Delta^*, \varepsilon^*, m^*, \eta^*)$ is also a bialgebra.

Example. Let G be a finite group. The dual of $\mathbb{K}G$ identifies with the algebra of maps $\mathbb{K}^G := \{f : G \to \mathbb{K}\}$. This algebra is a bialgebra with the coproduct given by $\Delta f(x \otimes y) = f(xy)$, for any $x, y \in G$. In particular, a basis of $\mathbb{K}^G$ is given by $\{\delta_x\}_{x \in G}$, where $\delta_x : G \to \mathbb{K}$, $y \mapsto \delta_{x,y}$. Then $\Delta(\delta_x)(y \otimes z) = \delta_{x,yz} = \left(\sum_{\substack{u,v \in G \\ uv = x}} \delta_u \otimes \delta_v\right)(y \otimes z)$.

Hence $\Delta(\delta_x) = \sum_{u \in G} \delta_u \otimes \delta_{u^{-1}x}$. The counit is given by $\varepsilon(\delta_x) = \delta_{x,e}$, with $e \in G$ the unit.

Definition Let B be a bialgebra and $I \subseteq B$ be a subspace.

i) We say that I is a <u>sub-bialgebra</u> of B if I is a subalgebra and a subcoalgebra.

ii) We say that I is a <u>bi-ideal</u> of B if I is an ideal and a coideal.

Proposition Let B be a bialgebra. For any bi-ideal I, B/I has a bialgebra structure induced by B.

Definition Let B, B' be bialgebras and $f : B \to B'$ be a linear map. We say that f is a <u>bialgebra morphism</u> if f is an algebra morphism and a coalgebra morphism.

Theorem Let B, B' be bialgebras and consider $f : B \to B'$ a bialgebra morphism. Then $Im(f)$ is a sub-bialgebra of B' and $ker(f)$ is a bi-ideal of B. Moreover, the bialgebras $B/Ker(f)$ and $Im(f)$ are isomorphic.

Proposition. Let V be a vector space. The tensor algebra $T(V)$ has a bialgebra structure defined by $\Delta(v) = 1 \otimes v + v \otimes 1$, $\forall v \in V$, which coincides with the unshuffle coproduct.

Proof. By universal property, there exists a well-defined algebra morphism $\Delta : T(V) \longrightarrow T(V) \otimes T(V)$ such that $\Delta(v) = 1 \otimes v + v \otimes 1$. This map is coassociative:
$$(\Delta \otimes id) \circ \Delta(v) = v \otimes 1 \otimes 1 + 1 \otimes v \otimes 1 + 1 \otimes 1 \otimes v = (id \otimes \Delta) \circ \Delta(v), \quad \forall v \in V.$$
Since $(\Delta \otimes id) \circ \Delta$, $(id \otimes \Delta) \circ \Delta : T(V) \longrightarrow T(V)^{\otimes 3}$ are algebra morphism (check), then we have that they are equal, so Δ is coassociative.

Analogously, we can define a unique algebra morphism $\varepsilon : T(V) \longrightarrow \mathbb{K}$ such that $\varepsilon(v) = 0$ for any $v \in V$. In particular $(\varepsilon \otimes id) \circ \Delta(v) = \varepsilon(v)1 + \varepsilon(1)v = v = (id \otimes \varepsilon) \circ \Delta$. Since $id, (id \otimes \varepsilon) \circ \Delta, (\varepsilon \otimes id) \circ \Delta$ are algebra morphism coincident on V, then they are the same. Therefore, since Δ and ε are algebra morphism, we conclude that $T(V)$ is a bialgebra. Furthermore, it is easy to see that $T(V)$ is cocommutative since $\tau \circ \Delta(v) = 1 \otimes v + v \otimes 1 = \Delta(v)$, $\forall v \in V$.

Finally, we will show that Δ is the unshuffle coproduct. By induction on the length of the words, n. If $n=1$, it is clear. Assume that the result holds for $n-1$. Hence
$$\Delta(v_1 \cdots v_n) = \Delta(v_1 \cdots v_{n-1}) \Delta(v_n) = \Big(\sum_{I \subseteq [n-1]} v_I \otimes v_{[n]\setminus I} \Big)(v_n \otimes 1 + 1 \otimes v_n)$$
$$= \sum_{I \subseteq [n-1]} v_I v_n \otimes v_{[n]\setminus I} + \sum_{I \subseteq [n-1]} v_I \otimes v_{[n]\setminus I} v_n$$
$$= \sum_{\substack{I \subseteq [n] \\ n \in I}} v_I \otimes v_{[n]\setminus I} + \sum_{\substack{I \subseteq [n] \\ n \notin I}} v_I \otimes v_{[n]\setminus I} = \sum_{I \subseteq [n]} v_I \otimes v_{[n]\setminus I}.$$
On the other hand, $\varepsilon(v_1 \cdots v_n) = \varepsilon(v_1) \cdots \varepsilon(v_n) = 0$. ∎

Analogously, we have:

Proposition Let V be a vector space. The symmetric algebra $S(V)$ has a bialgebra structure defined by $\Delta(v) = 1 \otimes v + v \otimes 1$, $\forall v \in V$. $S(V)$ is commutative and cocommutative.

Remark. In terms of polynomials, we have that $\mathbb{K}[X_1, \dots, X_n]$ has a bialgebra structure given by $\Delta(X_i) = 1 \otimes X_i + X_i \otimes 1$, for any $1 \leq i \leq n$. It is commutative and cocommutative.

Definition A <u>Lie algebra</u> is a vector space L together with a binary operation $[\cdot,\cdot] : L \times L \longrightarrow L$ called the Lie bracket, satisfying:

i) $[\cdot,\cdot]$ is bilinear; ii) $[x,x] = 0$, $\forall x \in L$;

ii) $[\cdot,\cdot]$ satisfies the Jacobi identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$, $\forall x,y,z \in L$.

Proposition. Let A be an associative algebra. Then $(A, [\cdot,\cdot])$ is a Lie algebra, where the Lie bracket is given by $[x,y] := xy - yx$, $\forall x,y \in A$.

In relation with bialgebras, we have the following distinguished elements.

Definition. Let B a bialgebra. An element $x \in B$ is called <u>primitive</u> if $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set of primitive elements is denoted $\mathrm{Prim}(B)$.

Proposition Let B be a bialgebra. Then $\mathrm{Prim}(B)$ is a Lie algebra for the Lie bracket $[x,y] = xy - yx$, $\forall x,y \in \mathrm{Prim}(B)$.

Proof. It is easy to see that $\mathrm{Prim}(B)$ is a vector subspace. Now, since Δ is an algebra morphism, we have for $x,y \in \mathrm{Prim}(B)$: $\Delta([x,y]) = \Delta(x)\Delta(y) - \Delta(y)\Delta(x) = (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)$
$= (xy - yx) \otimes 1 + 1 \otimes (xy - yx) = [x,y] \otimes 1 + 1 \otimes [x,y]$. Hence $[x,y] \in \mathrm{Prim}(B)$, i.e. $\mathrm{Prim}(B)$ is a Lie (sub-)algebra (of B). ∎

Convolution algebra

Let (A, m, η) and (C, Δ, ε) be an algebra and a coalgebra, respectively.

Definition The <u>convolution algebra</u> of C and A is the linear space $\mathrm{Hom}(C,A)$ with product defined by $f * g = m \circ (f \otimes g) \circ \Delta$ for all $f, g \in \mathrm{Hom}(C,A)$ and identity given by $\eta \circ \varepsilon$

Remark If $A = \mathbb{k}$ then $\mathrm{Hom}(C, \mathbb{k}) = C^*$. If $C = \mathbb{k}$, $\mathrm{Hom}(\mathbb{k}, A) = A$.

Lemma If $\pi : C \to D$ is a coalgebra morphism, then $\pi^* : \mathrm{Hom}(D,A) \to \mathrm{Hom}(C,A)$, $\pi^*(f) = f \circ \pi$ is an algebra morphism.

Proof. Notice for $f, g \in \mathrm{Hom}(D,A)$: $\pi^*(f * g) = (f * g) \circ \pi = m \circ (f \otimes g) \circ \Delta_D \circ \pi = m \circ (f \otimes g) \circ (\pi \otimes \pi) \circ \Delta_C$ $= \pi^*(f) * \pi^*(g)$. Also $\pi^*(\eta \circ \varepsilon_D) = \eta \circ \varepsilon_D \circ \pi = \eta \circ \varepsilon_C$. Hence π^* is an algebra morphism.

Proposition Let C be a bialgebra and A be an algebra. Suppose that $f \in \mathrm{Hom}(C,A)$ has a convolution inverse f^{-1}. Let A^{op} be the opposite algebra of A: $m_{A^{op}}(a \otimes b) = m(b \otimes a)$, $\forall a, b \in A$.

a) If $f : C \to A$ is an algebra map then $f^{-1} : C \to A^{op}$ is an algebra map.

b) If $f : C \to A^{op}$ is an algebra map then $f^{-1} : C \to A$ is an algebra map.

Proof. Let $D = C \otimes C$ be the tensor product coalgebra. Since C is a bialgebra, then $m_C : D \to C$ is a coalgebra morphism. Then by the previous lemma $m_C^*(f)$ has an inverse $m_C^*(f^{-1})$ in $\mathrm{Hom}(D,A)$. We will show that $\lambda : D \to A$, $\lambda(c \otimes d) = f^{-1}(d) f^{-1}(c)$ is a left convolution inverse for $m_C^*(f)$ as well. This implies that $f^{-1} \circ m_C = m_C^*(f^{-1}) = \lambda$.

Indeed, for $c, d \in C$ we have:

$$\left(\lambda * m_C^*(f)\right)(c \otimes d) = \sum_{(c),(d)} \lambda(c_{(1)} \otimes d_{(1)})\, m_C^*(f)\,(c_{(2)} \otimes d_{(2)})$$

$$= \sum_{(c),(d)} f^{-1}(d_{(1)})\, f^{-1}(c_{(1)})\, f(c_{(2)} d_{(2)})$$

f is an alg. morphism →
$$= \sum_{(c),(d)} f^{-1}(d_{(1)})\, \underbrace{f^{-1}(c_{(1)})\, f(c_{(2)})}_{=\; m_C \circ (f^{-1} \otimes f) \circ \Delta_C(c) \;=\; \eta(\varepsilon(c)) \;=\; \varepsilon(c) 1_C}\, f(d_{(2)})$$

$$= \sum_{(d)} f^{-1}(d_{(1)})\, \varepsilon(c)\, 1_C\, f(d_{(2)}) = \varepsilon(c)\varepsilon(d) 1_C = \varepsilon_D(c \otimes d) 1_C$$
$$= (\eta_C \circ \varepsilon_D)(c \otimes d)$$

then λ is a left convolution inverse for $m_C^*(f)$.

Lemma If $j : A \to B$ is an algebra morphism then $j_* : \mathrm{Hom}(C,A) \to \mathrm{Hom}(C,B)$ is an algebra morphism.

Proposition Let A be a bialgebra and C be a coalgebra. Suppose that $f \in \mathrm{Hom}(C,A)$ has a convolution inverse f^{-1}. Let $A^{cop} = (A, \tau_{C,C} \circ \Delta, \varepsilon)$ be the opposite coalgebra of A. $\tau_{C,C} : C \otimes C \to C \otimes C$, $c \otimes d \mapsto d \otimes c$

a) If $f : C \to A$ is a coalgebra morphism then $f^{-1} : C \to A^{cop}$ is a coalgebra morphism.

b) If $f : C \to A^{cop}$ is a coalgebra morphism then $f : C \to A$ is a coalgebra morphism.