RANDOM MATRICES WINTER 2024

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Exercise Set 1

Exercise 1. Simulate spectra of real random matrices $X = [x_{ij}]_{\substack{1\leq i\leq N\\ 1\leq j\leq M}}$ with different distributions in MATLAB, Mathematica or R, etc.

a) For $M = N$, describe the spectrum of X.
b) For $M = N$, describe the spectrum of $(X + X^T)/2$.
c) For $M \neq N$, describe the spectrum of $X^T X$ for different rations of M/N.

Consider $M, N \sim 10, 100, 1000$.

Exercise 2. Let $\vec{X} = (X_1, X_2, \dots, X_N)$ be a Gaussian standard vector, i.e. $X_1, \dots, X_N$ are independent identically distributed random variables $\mathcal{N}(0, 1)$.

a) Show that for the random variables $Y = \sum a_i X_i$ and $Z = \sum b_i X_i$, we have
$$Y \perp\!\!\!\perp Z \quad \Leftrightarrow \quad \mathbb{E}[\,YZ\,] = 0 \quad \Leftrightarrow \quad a \perp b,$$
where $a = (a_1, \dots, a_N)$ and $b = (b_1, \dots, b_N)$.
b) Let U be an $N \times N$ orthogonal matrix. Show that $\vec{Y} = U\vec{X}$ is also a Gaussian standard vector.

Exercise 3. Let X_N be an element in the *Gaussian Orthogonal Ensemble* GOE(N), i.e. $X_N = [x_{ij}]_{i,j=1}^N$ is a random matrix where $x_{ij} = x_{ji}$ for all i, j, and x_{ij} $(1 \leq i \leq j \leq N)$ are independent real random variables with Gaussian distribution of mean zero, $\mathbb{E}[x_{ii}^2] = 2/N$ and $\mathbb{E}[x_{ij}^2] = 1/N$ if $i \neq j$. Show that if O is an $N \times N$ orthogonal matrix, then OX_NO^T is in GOE(N).

Exercise 4 (Concentration around equator)**.** Let B_p be the unit ball in $\mathbb{R}^p$. The aim of this exercise is to prove that

$$\mathbb{P}\big((t_1, \dots, t_p) \in B_p \,:\, |t_p| \geq \epsilon\big) \leq \sqrt{2\pi} \exp\left(-\epsilon^2 \frac{p-1}{2}\right), \quad \text{for } \ p \geq 3. \tag{0.1}$$

a) Prove for $y \geq 0$
$$\int_y^\infty \exp(-t^2)\,\mathrm{d}t \leq \frac{\sqrt{\pi}}{2} \exp(-y^2).$$
b) Let $p \geq 3$. Prove that
$$\int_0^1 (1-t^2)^{\frac{p-1}{2}} \mathrm{d}t \geq \int_0^{\frac{1}{\sqrt{p-1}}} (1-t^2)^{\frac{p-1}{2}} \mathrm{d}t \geq \frac{1}{2\sqrt{p-1}}.$$
Hint: Bernoulli's inequality states that $(1+a)^b \geq 1 + ab$ *for all real numbers* $b \geq 1$ *and* $a \geq -1$.
c) Prove that if $p \geq 1$ and $0 < \epsilon \leq 1$, then $(1-\epsilon)^p \leq \exp(-\epsilon p)$.
d) Use c) to prove (0.1). *Hint: Recall that*
$$\mathbb{P}\big((t_1, \dots, t_p) \in B_p \,:\, |t_p| \geq \epsilon\big) = 2\frac{\mathrm{vol}(B_{p-1})}{\mathrm{vol}(B_p)} \int_\epsilon^1 (1-t^2)^{\frac{p-1}{2}} \mathrm{d}t.$$

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e) Conclude that two independent random vectors in B_p are almost orthogonal with high probability.

Exercise 5 (The lattice of set partitions)**.** Let $\mathcal{P}(n)$ be the set of partitions of $\{1, \dots, n\}$.

a) Let $\pi, \rho \in \mathcal{P}(n)$. Show that the sets

$$\{\sigma \in \mathcal{P}(n) \,:\, \pi \leq \sigma \text{ and } \rho \leq \sigma\} \qquad \text{and} \qquad \{\sigma \in \mathcal{P}(n) \,:\, \pi \geq \sigma \text{ and } \rho \geq \sigma\}$$

have unique minimal and maximal element denoted by $\pi \vee \rho$ and $\pi \wedge \rho$, respectively.

b) For any $\pi, \rho \in \mathcal{P}(n)$, describe $\pi \vee \rho$ and $\pi \wedge \rho$.

c) Draw the Hasse diagram of $\mathcal{P}(4)$.