RANDOM MATRICES WINTER 2024

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Exercise Set 2

Exercise 1 (Hafnian and Pfaffian)**.** Let A be a symmetric resp. antisymmetric $2n \times 2n$-matrix. The numbers

$$\operatorname{haf}(A) = \sum_{\pi \in \mathcal{P}_2(n)} \prod_{\{i,j\} \in \pi} a_{ij} \qquad \text{resp.} \qquad \operatorname{pf}(A) = \sum_{\pi \in \mathcal{P}_2(n)} \operatorname{sgn}(\pi) \prod_{\{i,j\} \in \pi} a_{ij}$$

are called the *Hafnian* resp. *Pfaffian* of A. Here the signature of a matching π is defined as follows: If $\pi = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ with $i_k < j_k$ and $i_1 < i_2 < \cdots < i_n$, then $\operatorname{sgn}(\pi) = \operatorname{sgn}(\sigma_\pi)$, where σ_π is the permutation such that $\sigma_\pi(2k-1) = i_k$ and $\sigma_\pi(2k) = j_k$, i.e.,

$$\operatorname{sgn}(\pi) = \operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & i_n & j_n \end{pmatrix}$$

(1) Show that for a symmetric resp. antisymmetric matrix A

$$\operatorname{haf}(A) = \frac{1}{n!2^n} \sum_{\sigma \in \mathcal{S}_{2n}} \prod_{i=1}^n a_{\sigma(2i-1)\sigma(2i)}. \qquad \operatorname{pf}(A) = \frac{1}{n!2^n} \sum_{\sigma \in \mathcal{S}_{2n}} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(2i-1)\sigma(2i)}.$$

(2) Show that for any $n \times n$ matrix C,

$$\operatorname{per}(C) = \operatorname{haf} \begin{pmatrix} 0 & C \\ C^T & 0 \end{pmatrix} \qquad \text{and} \qquad \det(C) = (-1)^{n(n-1)/2} \operatorname{pf} \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}$$

where

$$\operatorname{per}(C) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^n a_{i\sigma(i)}$$

is the *permanent* of a matrix C.

Exercise 2 (Complex Isserlis-Wick formula)**.**

(1) Prove the complex Isserlis-Wick formula: let $Z_1, \dots, Z_p$ be independent standard complex Gaussian random variables and consider $Y_1, \dots, Y_n \in \{Z_1, \bar{Z}_1, \dots, Z_p, \bar{Z}_p\}$. Then

$$\begin{aligned} \mathbb{E}(Y_1 \cdots Y_n) &= \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \text{connecting } Z_i \text{ with } \bar{Z}_i}} \mathbb{E}_\pi(Y_1, \dots, Y_n) \\ &= \left| \{\text{pairings of } \{1, \dots, n\} \text{ which connect } Z_i \text{ with } \bar{Z}_i\} \right|. \end{aligned}$$

(2) Show that if Z is a standard complex Gaussian random variable, then $\mathbb{E}(Z^n \bar{Z}^m) = \delta_{n,m} n!$.

Exercise 3 (Stein's Lemma)**.**

(1) Let X and Y be two jointly normally distributed random variables and g be a differentiable function. Show that

$$\operatorname{Cov}(g(X), Y) = \operatorname{Cov}(X, Y) \mathbb{E}(g'(X)).$$

(2) If $(X_1, \dots, X_n) \sim \mathcal{N}(\mu, \Sigma)$, show that

$$\mathbb{E}\big(g(X)(X - \mu)\big) = \Sigma \cdot \mathbb{E}\big(\nabla g(X)\big).$$

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Exercise 4. Recall that a standard real Gaussian random matrix GOE(N) is a random matrix of the form

$$X_N = \frac{1}{\sqrt{N}}\left(\frac{A+A^T}{2}\right),$$

where $A = (a_{ij})_{i,j=1}^N$ is a random matrix with where a_{ij} are i.i.d. with distribution $\mathcal{N}(0,1)$. Similarly, a standard Hermitian complex Gaussian random matrix GUE(N) is a random matrix of the form

$$X_N = \frac{1}{\sqrt{N}}\left(\frac{(A+\mathbf{i}B)+(A+\mathbf{i}B)^*}{2}\right),$$

where $A = (a_{ij})_{i,j=1}^N$ and $B = (b_{ij})_{i,j=1}^N$ are independent random matrices with where $a_{ij}, b_{i,j}$ are i.i.d. with distribution $\mathcal{N}(0,1)$.

(1) Let $X_N = (x_{ij}) \sim$ GOE(N). Compute $\mathbb{E}(x_{ii}^2), \mathbb{E}(x_{ij}^2)$ and $\mathbb{E}(x_{ij}x_{kl})$.
(2) Let $X_N = (x_{ij}) \sim$ GUE(N). Compute $\mathbb{E}(|x_{ij}|^2)$ and $\mathbb{E}(x_{ij}x_{kl})$.
(3) Show that GUE is invariant under unitary conjugation, i.e. if $X_N \sim$ GUE(N) then $UX_NU^* \sim$ GUE(N), for any U unitary $N \times N$ matrix.

Exercise 5.
(1) Let $X_N \sim$ GUE(N) and $A_i \in \mathbb{C}^{N\times N}$ for $i = 1,2,3,4$. Compute

$$\mathbb{E}\frac{1}{N}\operatorname{Tr}\left(A_1X_NA_2X_N\right) \qquad \text{and} \qquad \mathbb{E}\frac{1}{N}\operatorname{Tr}(X_NA_1X_NA_2X_NA_3X_NA_4)$$

in terms of traces of products of A_i's.
(2) Let X_N, Y_N independent random matrices in GUE(N). Compare the asymptotic behavior of $\mathbb{E}\frac{1}{N}\operatorname{Tr}(X_NY_NX_NY_N)$ and $\mathbb{E}\frac{1}{N}\operatorname{Tr}(X_N^2Y_N^2)$ as $N \to \infty$.

Exercise 6. Let $A = (a_{ij})_{i,j=1}^N$ be a GUE(N) random matrix with entries $a_{ii} = x_{ii}$ and $a_{ij} = x_{ij} + \mathbf{i}y_{ij}$. Consider the N^2 random vector

$$(x_{11}, \ldots, x_{NN}, x_{1N}, \ldots, x_{N-1N}, y_{12}, \ldots, y_{1N}, \ldots, y_{N-1N}).$$

Show that the random vector has the density

$$C\exp\left(-N\frac{\operatorname{Tr}(A^2)}{2}\right)\mathrm{d}A,$$

where C is a constant and

$$\mathrm{d}A = \prod_{i=1}^N \mathrm{d}x_{ii}\prod_{i<j}\mathrm{d}x_{ij}\mathrm{d}y_{ij}.$$

Evaluate the constant C.

Exercise 7. We say that a partition $\pi \in \mathcal{P}(n)$ has *type* $(r_1, \ldots, r_n)$ if it has r_i blocks of size i. Show that the number of partitions of $[n]$ of type $(r_1, r_2 \ldots, r_n)$ is

$$\frac{n!}{(1!)^{r_1}(2!)^{r_2}\cdots(n!)^{r_n}r_1!r_2!\cdots r_n!}.$$