RANDOM MATRICES WINTER 2024

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Exercise Set 3

Exercise 1. Let $X_N \sim \mathrm{GUE}(N)$. Compute $\mathbb{E}\frac{1}{N}\operatorname{Tr}\left(X_N^6\right)$ by using:

a) Isserlis–Wick formula;
b) cycles of $\gamma \circ \pi$, where $\gamma = (1\ 2\ 3\ \cdots\ m) \in S_m$;
c) genus expansion.

Exercise 2. Find explicit bijections between the following sets:

a) $\mathrm{NC}(n)$,
b) $\mathrm{NC}_2(2n)$,
c) planar rooted trees with $n+1$ vertices,
d) Dyck paths of length $2n$,
e) Triangulations of an $(n+2)$-gon,
f) Parenthesizations of product $x_0 \cdot x_1 \cdot \ldots \cdot x_n$.

Also, try to realize the Catalan recurrence $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$ in each example.

Exercise 3. Let S_n be the symmetric group on $[n]$ and consider its Coxeter generator subset given by the transpositions of the form $s_i = (i \quad i+1)$ for $1 \leq i < n$. On the other hand, the number of *crossings* of $\sigma \in S_n$ is defined as follows: In a $2 \times m$ matrix, write the numbers $1, 2, \ldots, n$ in the first row, and the numbers $\sigma(1), \sigma(2), \ldots, \sigma(n)$ in the second row. Then for each $1 \leq i \leq n$, draw a line between i in the first row and i in the second row. The number of crossings of σ, denoted by $\mathrm{cr}(\sigma)$, is given by the the number of crossings between the lines in the draw. For instance

$$\sigma \quad \sim \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \\ 5 & 3 & 1 & 4 & 6 & 2 \end{pmatrix} \qquad \to \qquad \mathrm{cr}(\sigma) = 8.$$

If $\ell(\sigma) = \min\{k \,:\, \sigma = s_{i_1} \cdots s_{i_k}\}$ stands for the length of σ, show that $\ell(\sigma) = \mathrm{cr}(\sigma)$.

Exercise 4 (The lattice of non-crossing partitions)**.** Let $\mathrm{NC}(n)$ be the set of partitions of $\{1, \ldots, n\}$.

a) Let $\pi, \rho \in \mathrm{NC}(n)$. Show that the sets
$$\{\sigma \in \mathrm{NC}(n) \,:\, \pi \leq \sigma \text{ and } \rho \leq \sigma\} \qquad \text{and} \qquad \{\sigma \in \mathrm{NC}(n) \,:\, \pi \geq \sigma \text{ and } \rho \geq \sigma\}$$
have unique minimal and maximal element denoted by $\pi \vee \rho$ and $\pi \wedge \rho$, respectively.
b) Show that $\mathrm{NC}(n)$ is a lattice.
c) Draw the Hasse diagram of the interval $\left[0_6, \left\{\{1,4,5,6\},\{2,3\}\right\}\right]$.

Exercise 5. Consider the Cayley graph of (S_n, E), where S_n stands for the symmetric group on $[n]$ and $E \subset S_n$ is the set of all transpositions of S_n. Define the map $\iota : \mathcal{P}(n) \to S_n$ as follows.

i) If $V \subset [n]$ where $V = \{i_1, i_2 \ldots, i_s\}$ is such that $i_1 < i_2 < \cdots < i_s$, then $\iota(V)$ is the cycle $(i_1\ i_2\ \cdots\ i_s)$;

Date: October 25, 2024.

ii) If $\pi = \{V_1, \dots, V_k\}$ then $\iota(\pi) = \iota(V_1) \cdots \iota(V_k)$. Observe that the product is well-defined since the cycles $\iota(V_i)$ are disjoint and, hence, mutually commuting.

Denote by $e \in S_n$ the identity element and $c = (1\ 2\ \cdots\ n) \in S_n$. Prove that ι induces a bijection between $\mathrm{NC}(n)$ and $[e, c]$, the set of $\sigma \in S_n$ such that there is a geodesic $(\tau_0, \dots, \tau_m)$ from $\tau_0 = e$ to $\tau_m = c$ with $\sigma = \tau_i$ for some i.

Hint. The following may be helpful: for $t_1, t_2 \in S_n$, we have that t_2 covers t_1 in the Cayley graph if and only if $t_2 = t_1 r$, where $r = (i\ j)$ is a transposition such that i and j belong to different orbits of t_1. The effect of the right multiplication with r in the equality $t_2 = t_1 r$ is that the two orbits of t_1 which contain i and j are united into one orbit of t_2 (which thus contains both i and j).

Bonus. Prove that ι restricted to $\mathrm{NC}(n)$ is order-preserving (hence a poset isomorphism).