RANDOM MATRICES WINTER 2024

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Exercise Set 4

Exercise 1. (1) Let a and b be real numbers with $b \leq 0$. Let $G(z) = (z - a - \mathbf{i}\, b)^{-1}$. Show that G is the Cauchy transform of a probability measure, $\delta_{a+\mathbf{i}\, b}$, which has a density and find its density using Stieltjes inversion.

(2) Let ν be a probability measure on $\mathbb{R}$ with Cauchy transform G. Show that $\tilde{G}$ the Cauchy transform of $\delta_{a+\mathbf{i}\, b} * \nu$ is the function $\tilde{G}(z) = G(z - (a + \mathbf{i}\, b))$. Here $*$ denotes the classical convolution of probability measures.

Exercise 2. (1) Determine the Cauchy transformation and the corresponding measure for the moment sequence given by $m_{2n} = \binom{2n}{n}$, and $m_{2n+1} = 0$, for any $n \geq 0$.

(2) Determine the corresponding measure for the following Cauchy transform:

$$G(z) = \frac{z + 1 - c - \sqrt{(z-a)(z-b)}}{2z},$$

where $0 < c < \infty$ is a positive real number, $a = (1 - \sqrt{c})^2$ and $b = (1 + \sqrt{c})^2$.

Exercise 3. (1) Let $f : \mathbb{C}^+ \to \mathbb{C}^+$ be the Stieltjes transform of a probability measure on $\mathbb{R}$. Show that the matrix

$$\left[\frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z_j}} \right]_{i,j=1}^{n}$$

is positive semidefinite, for any $z_1, \ldots, z_n \in \mathbb{C}^+$, $n \in \mathbb{N}$.

(2) If

$$\left[\frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z_j}} \right]_{i,j=1}^{2}$$

is positive definite for any $z_1, z_2 \in \mathbb{C}^+$, show that $f(\mathbb{C}^+) \subseteq \mathbb{C}^+$. Find an example in which f is not analytic.

Exercise 4. Recall that if $\varphi : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic, the Nevanlinna representation theorem asserts that there is a unique finite positive Borel measure σ on $\mathbb{R}$ and real numbers a, b with $b \geq 0$ such that for any $z \in \mathbb{C}^+$:

$$\varphi(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} \, \mathrm{d}\sigma(t).$$

Now, suppose that $G : \mathbb{C}^+ \to \mathbb{C}^-$ is analytic and in addition $\limsup_{y \to \infty} y |G(\mathbf{i}\, y)| = c < \infty$. Show that there is a unique positive Borel measure ν on $\mathbb{R}$ such that

$$G(z) = \int_{\mathbb{R}} \frac{1}{z - t} \, \mathrm{d}\nu(t) \quad \text{and} \quad \nu(\mathbb{R}) = c.$$

Hint. From the Nevanlinna representation applied to $-G$, prove that $b = 0$, σ has finite second moment, and $a = -\int_{\mathbb{R}} t \, \mathrm{d}\sigma(t)$.

Exercise 5. Let ν be a probability measure on $\mathbb{R}$ and $\theta > 0$. In this exercise, we will consider limits as $z \to \infty$ in a Stolz angle $S_\theta = \{x + \mathbf{i}\, y : \theta y > |x|\}$. Show that

(1) for $z \in S_\theta$ and $t \in \mathbb{R}$, $|z - t| \geq |t| / \sqrt{1 + \theta^2}$;

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(2) for $z \in S_\theta$ and $t \in \mathbb{R}$, $|z-t| \geq |z|/\sqrt{1+\theta^2}$;

(3) $\displaystyle\lim_{z\to\infty} \int_{\mathbb{R}} \frac{t}{z-t}\, \mathrm{d}\nu(t) = 0$;

(4) $\displaystyle\lim_{z\to\infty} zG(z) = 1$.

Note. Given $\theta > 0$ and f a function on S_θ, recall that f *converges to c in a Stolz angle S_θ* if for any $\epsilon > 0$ there is $\beta > 0$ such that $|f(z) - c| < \epsilon$ for any $z \in S_{\theta,\beta} = \{z \in S_\theta : \mathrm{Im}(z) > \beta\}$.

Exercise 6. Let ν be a probability measure on $\mathbb{R}$. Suppose ν has absolute moments up to order n, i.e. $\int_{\mathbb{R}} |t|^n \,\mathrm{d}\nu(t) < \infty$. Let $\alpha_1, \ldots, \alpha_n$ be the first n moments of ν, i.e. $\alpha_k = \int_{\mathbb{R}} t^k \,\mathrm{d}\nu(t)$, for $1 \leq k \leq n$. Let $\theta > 0$ be given. In the following, all limits as $z \to \infty$ will be assumed to be in a Stolz angle $S_\theta = \{x + \mathbf{i}\, y : \theta y > |x|\}$.

(1) Show that

$$\lim_{z\to\infty} \int_{\mathbb{R}} \left| \frac{t^{n+1}}{z-t} \right| \mathrm{d}\nu(t) = 0.$$

(2) Show that

$$\lim_{z\to\infty} z^{n+1} \left(G(z) - \left(\frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_2}{z^3} + \cdots + \frac{\alpha_n}{z^{n+1}} \right) \right) = 0.$$

Exercise 7. Suppose that $\theta > 0$ and ν is a probability measure on $\mathbb{R}$ and that for some $n > 0$ there are real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ such that as $z \to \infty$ in S_θ

$$\lim_{z\to\infty} z^{n+1} \left(G(z) - \left(\frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{\alpha_2}{z^3} + \cdots + \frac{\alpha_{2n}}{z^{2n+1}} \right) \right) = 0.$$

Show that ν has moments up to order $2n$, i.e. $\int_{\mathbb{R}} t^{2n} \,\mathrm{d}\nu(t) < \infty$ and that $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ are the first $2n$ moments of ν.