RANDOM MATRICES WINTER 2024

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Exercise Set 5

Exercise 1. Let $\{m_n\}_{n\geq 0}$ be a sequence of moments of a probability measure μ and for $n \geq 1$ define

$$\Delta_n = \begin{vmatrix} 1 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & & & \\ m_2 & & \ddots & & \vdots \\ \vdots & & & & \\ m_n & m_{n+1} & & \cdots & m_{2n} \end{vmatrix}$$

the corresponding Hankel determinant. Show

(1) The orthogonal polynomials of μ are given by

$$p_n(x) = \frac{1}{\sqrt{\Delta_{n-1}\Delta_n}} \begin{vmatrix} 1 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & & & \\ m_2 & & \ddots & & \vdots \\ \vdots & & & & \\ m_{n-1} & m_n & & \cdots & m_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}.$$

(2) $b_n = \dfrac{\sqrt{\Delta_{n-1}\Delta_{n+1}}}{\Delta_n}$ for any $n \geq 1$.

(3) The monic orthogonal polynomials $\{\tilde{p}_n\}_{n\geq 0}$ satisfy the following recursion:

$$x\tilde{p}_n = \tilde{p}_{n+1}(x) + a_n\tilde{p}_n(x) + b_{n-1}^2\tilde{p}_{n-1}(n).$$

Exercise 2. Let μ be a probability measure and $\{p_n(x)\}_{n\geq 0}$ be the corresponding orthogonal polynomials with Jacobi parameters

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1}p_{n-1}(x) \tag{0.1}$$

with $p_0(x) = 1$ and $p_1(x) = \frac{x-a_0}{b_0}$. We set

$$\begin{aligned} K_n(x,y) &= \sum_{k=0}^{n} p_n(x)p_n(y) \\ G_n(z) &= \cfrac{1}{z - a_0 - \cfrac{b_0^2}{z - a_1 - \cfrac{b_1^2}{z - a_2 - \cfrac{b_2^2}{\ddots \cfrac{\ddots}{z - a_{n-1}}}}}} \\ q_n(x) &= \int \partial P_n(x,y)\,\mathrm{d}\mu(y) := \int \frac{p_n(x) - p_n(y)}{x-y}\,\mathrm{d}\mu(y). \end{aligned}$$

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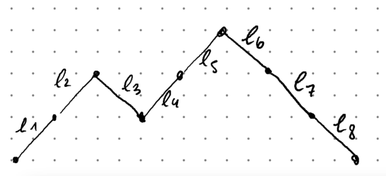
(1) Show that
$$K_n(x,y) = b_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}.$$
(2) Show that the sequence $\{q_n(x)\}_{n\geq 0}$ also satisfies the recurrence (0.1) for $n \geq 1$ with initial conditions $q_0(x) = 0$ and $q_1(x) = \frac{1}{b_0}$.

(3) Show that for any $n \geq 1$, we have that $G_n(z) = \dfrac{q_n(z)}{p_n(z)}$.

Exercise 3. Let $(y_0, y_1, \ldots, y_{2n})$ be a Dyck path. A *labelling* with values in a set L is a sequence $(l_1, l_2, \ldots, l_{2n})$ such that $l_i \in L$ for $1 \leq i \leq 2n$:



(1) Find a bijection between the complete matchings of the set $\{1, 2, \ldots, 2n\}$ and the labelled Dyck paths of length $2n$ and labelling $l_i \in L = \{1, 2, \ldots, n\}$, satisfying the following conditions:
$$\begin{cases} l_i = 1 & \text{when } y_{i-1} < y_i \text{ (up step)} \\ 1 \leq l_i \leq y_{i-1} & \text{when } y_{i-1} > y_i \text{ (down step)} \end{cases}$$
(2) Conclude that the Jacobi parameters of the standard normal distribution are given by $a_n = 0$ and $b_n^2 = n+1$ for any $n \geq 1$.

(3) Show that the orthogonal polynomials are given by
$$p_n(x) = c_n \left(x - \frac{\mathrm{d}}{\mathrm{d}x} \right)^n (1) = (-1)^n c_n e^{x^2/2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2/2},$$
where $c_n = \frac{1}{\sqrt{n!}}$.

Exercise 4. Let μ, μ_n be probability measures on $\mathbb{R}$, for any $n \geq 1$, with distribution functions F, F_n, respectively. Recall that $F(x) = \mu\big((\infty, x]\big)$ and $F_n(x) = \mu_n\big((\infty, x]\big)$, for any $n \geq 1$. Show the following:

(1) $\{\mu_n\}_{n\geq 1}$ converges vaguely to μ if and only if for every finite interval of continuity $I = [a,b]$ of F, we have that $\lim\limits_{n\to\infty} \big(F_n(b) - F_n(a)\big) = F(b) - F(a)$.

(2) $\{\mu_n\}_{n\geq 1}$ converges weakly to μ if and only if $\lim\limits_{n\to\infty} F_n(x) = F(x)$ for any $x \in \mathbb{R}$ where F is continuous.

Note. Let F be a distribution function. Recall that $I = [a,b]$ is an *interval of continuity* of F if F is continuous at the endpoints a and b.

Definition. Let μ, μ_n be probability measures on $\mathbb{R}$, for any $n \geq 1$.

(1) We say that μ_n *converges vaguely to* μ if
$$\lim_{n\to\infty} \int_{\mathbb{R}} f \, \mathrm{d}\mu_n = \int_{\mathbb{R}} f \, \mathrm{d}\mu, \qquad \text{for any } f \in \mathcal{C}_0(\mathbb{R}).$$
(2) We say that μ_n *converges weakly to* μ if
$$\lim_{n\to\infty} \int_{\mathbb{R}} f \, \mathrm{d}\mu_n = \int_{\mathbb{R}} f \, \mathrm{d}\mu, \qquad \text{for any } f \in \mathcal{C}_b(\mathbb{R}).$$