RANDOM MATRICES WINTER 2024

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Exercise Set 6

The goal of the present exercise set is to provide an analytic proof that the diagonal elements of a Wigner matrix do not play a role.

Exercise 1. For a matrix $A \in \mathbb{C}^{N\times N}$, the **spectral norm** $\| \, \|$ and the **Frobenius norm** $\| \, \|_2$ are defined by

$$\|A\| := \sup_{x\in\mathbb{C}^N} \frac{\|Ax\|_2}{\|x\|_2} \qquad\qquad \|A\|_2 := \operatorname{Tr}(A^*A)^{1/2},$$

where $\| \, \|_2$ denotes the Euclidean norm on $\mathbb{C}^N$. Furthermore, we use the spectral order: for $A, B \in \mathbb{C}^{N\times N}$ selfadjoint, we define

$$A \leq B \Leftrightarrow B - A \text{ is positive semidefinite.}$$

Show:

(1) $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$
(2) $\|A\| \leq 1 \Rightarrow A^*A \leq I$
(3) For A, B selfadjoint, $A \leq B \Rightarrow \operatorname{Tr}(A) \leq \operatorname{Tr}(B)$
(4) $|\operatorname{Tr}(AB)| \leq \|A\|_2\|B\|_2$
(5) $\|AB\|_2 \leq \|A\|\|B\|_2$
(6) $\|A\|_2 \leq \sqrt{N}\|A\|$
(7) If A is selfadjoint and $R_A(z) = (zI - A)^{-1}$, $z \in \mathbb{C}^+$, stands for the resolvent of A, then $\|R_A(z)\| \leq \dfrac{1}{\operatorname{Im} z}$.

Exercise 2. Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{E}(X_i) = 0$, $\mathbb{E}(|X_i|^2) = 1$, and $\mathbb{E}(|X_i|^4) < \infty$. Show that for the vector $Y_N = (X_1, X_2, \ldots, X_N)$ we have that

$$\sqrt{N} - \frac{\sigma^2}{2\sqrt{N}} \leq \mathbb{E}(\|Y_N\|_2) \leq \sqrt{N},$$

where $\sigma^2 = \frac{1}{N^2}\operatorname{Var}\left(\|Y_n\|_2^2\right)$.
Hint. The following inequality may be helpful:

$$\frac{1 + u - (u-1)^2}{2} \leq \sqrt{u} \leq \frac{1+u}{2}.$$

Exercise 3. For selfadjoint matrices A, B, show the inequality

$$\frac{1}{N}\left|\operatorname{Tr}\left(R_A(z) - R_B(z)\right)\right| \leq \frac{1}{\sqrt{N}}\frac{1}{(\operatorname{Im} z)^2}\|A - B\|_2,$$

and conclude that for Wigner matrices $X_N = \frac{1}{\sqrt{N}}[x_{ij}]$ with $\mathbb{E}(x_{ij}^4) < \infty$, the diagonal can be removed without chaning the limit:

$$\lim_{N\to\infty}\left|\mathbb{E}\frac{1}{N}\operatorname{Tr}(R_{X_N}(z)) - \mathbb{E}\frac{1}{N}\operatorname{Tr}(R_{\mathring{X}_N}(z))\right| = 0$$

where $\mathring{X}_N = X_N - \operatorname{diag} X_N$.

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