RANDOM MATRICES WINTER 2024

FRANZ LEHNER AND ADRIAN CELESTINO

Exercise Set 7

Exercise 1. Let $\mathcal{H}$ be a Hilbert space, $\psi \in \mathcal{H}$ be a unit vector ("state") and $A \in B(\mathcal{H})$ be a self-adjoint operator ("observable").

(1) Show that the numbers
$$m_n(A) = \langle A^n \psi, \psi \rangle$$
form the sequence of moments of a probability measure with Cauchy transform
$$G_A(z) = \langle (zI - A)^{-1} \psi, \psi \rangle.$$
The corresponding measure is called the distribution of A with respect to the state vector ψ.

(2) We form the orthogonal decomposition $\mathcal{H} = \mathbb{C}\psi \oplus \mathring{\mathcal{H}}$ and decompose the operator accordingly into a block matrix
$$A = \begin{bmatrix} \alpha & b^* \\ b & \mathring{A}, \end{bmatrix}$$
where $\alpha \in \mathbb{R}$ and $b \in \mathring{\mathcal{H}}$. Show that
$$G_A(z) = \bigl(z - \alpha - b^*(z - \mathring{A})^{-1} b\bigr)^{-1}.$$

Exercise 2. (1) Let $\mathcal{H} = \ell^2(\mathbb{N}_0)$ with the canonical basis $\{e_i\}_{i \geq 0}$ and $S : \mathcal{H} \to \mathcal{H}$ be the shift operator, i.e. $S(e_i) = e_{i+1}$. Show that the distribution of the operator $A = S + S^*$ with respect to the state vector $\psi = e_0$ is given by the semicircular distribution.

(2) Show that the moments of a standard normal distribution are given by
$$[x^0] \left(x + \frac{\mathrm{d}}{\mathrm{d}x} \right)^n 1,$$
where 1 denotes the constant polynomial and $[x^0]p(x)$ denotes the constant coefficient of a polynomial $p(x)$.

Exercise 3. If we interpret $\mathbb{N}_0$ as the set of all words over an alphabet with exactly one element, we can generalize the previous example as follows: we consider the Hilbert space $\mathcal{H}$ with orthonormal basis $\{e_w : w \in \{1,2\}^*\}$, where
$$\{1,2\}^* = \bigl\{ i_1 i_2 \cdots i_n : n \in \mathbb{N},\, i_j \in \{1,2\} \bigr\} \cup \{\varepsilon\}$$
is the set of all wods over the alphabet $\{1,2\}$ including the empty word ε. The shift and its adjoint can now be interpreted as the creation and annihilation of letters:
$$S_1 e_w = e_{1w}, \qquad S_2 e_w = e_{2w},$$
i.e. 1 or 2 is added to the front of the word. Again, we use $\psi = e_\varepsilon$ to denote the state of the empty word ("vacuum"). Show:

Date: January 10, 2025.

(1) The adjoints are given by

$$\begin{array}{ccc} S_1^*\psi = 0 & S_1^*e_{1w} = e_w & S_1^*e_{2w} = 0 \\ S_2^*\psi = 0 & S_2^*e_{1w} = 0 & S_2^*e_{2w} = e_w \end{array}$$

and the distribution of both $X_1 = S_1 + S_1^*$ and $X_2 = S_2 + S_2^*$ with respect to the vacuum state is the semicircular distribution.

(2) The distribution of $X_1 + X_2$ with respect to the vacuum state is the semicircular distribution with variance 2.

(3) The set of linear combinations

$$c_0 + \sum_{m,n\geq 0} c_{m,n} S_1^m (S_1^*)^n$$

forms a subalgebra $\mathcal{A}_1$ of $B(\mathcal{H})$ (analogously for $\mathcal{A}_2$).

(4) We have that

$$\langle A_1 A_2 \cdots A_n \psi, \psi \rangle = 0$$

if the following conditions hold:
- $A_j \in \mathcal{A}_{i_j}$, where $i_j \neq i_{j+1}$ for every $j = 1, \ldots, n-1$.
- $\langle A_j \psi, \psi \rangle = 0$ for every $j = 1, \ldots, n$.

Exercise 4. For a real random variable X with all its moments $\mathbb{E}|X|^n < \infty$, we define

$$\mathcal{F}_X(z) = \mathbb{E}e^{zX} = \sum_{n=0}^{\infty} \frac{m_n(X)}{n!} z^n$$

the (formal) Laplace transform of X. The *cumulants* $\kappa_n(X)$ of X are defined by the generating function

$$\mathcal{L}_X(z) = \log \mathcal{F}_X(z) = \sum_{n=1}^{\infty} \frac{\kappa_n(X)}{n!} z^n.$$

(1) Show that the cumulants satisfy the following properties:
   (a) $\kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y)$ if X and Y are independent.
   (b) $\kappa_n(\lambda X) = \lambda^n \kappa_n(X)$.
   (c) There are universal polynomials $P_n(x_1, x_2, \ldots, x_{n-1})$ such that $P_n(0, 0, \ldots, 0) = 0$ and
   $$m_n(X) = \kappa_n(X) + P_n\big(\kappa_1(X), \kappa_2(X), \ldots, \kappa_{n-1}(X)\big).$$

(2) Let $X_1, X_2, \ldots, X_n$ be i.i.d. copies of X and $\omega = e^{2\pi i/n}$ be a primitive root of unity. Show that

$$\kappa_n(X) = \frac{1}{n} \mathbb{E}\left[\big(\omega X_1 + \omega^2 X_2 + \cdots + \omega^n X_n\big)^n\right].$$