Problem sheet 2
Jan 20th 2004

MT290 Complex variable

Ex. 1
Find the derivatives $\frac{df}{dz}$ (with some explanation) and show that the Cauchy-Riemann equations are satisfied for the functions $f_1(z) = z^4$ and $f_2(z) = e^{2z}$.

Ex. 2
Let $f(x+iy) = x^2 + 2ixy$. Writing $f = u(x,y) + iv(x,y)$ find the partial differentials of u and v. Find where the Cauchy-Riemann equations are satisfied and hence find where this function is differentiable (the answer is a simple line).

Ex. 3
Let $f(x+iy) = y(1-x^2) - iy^2x$. Find the set of points where this function is differentiable and sketch the set on an Argand diagram. Do the real and imaginary parts of this function satisfy Laplace's equation ?

Ex. 4
Let $z = x + iy, x > 0, y > 0$. Express $\log|z|$ and $\arg z$ as functions of x and y (so don't use i: $\log|z| = \log|x+iy|$ won't do !). Thus show that the function: $f(z) = \log|z| + i\arg z$ satisfies the Cauchy-Riemann equations for $x, y > 0$. Since the derivatives are continuous in this region that means that $f(z)$ is differentiable. Show that $f'(z) = \frac{1}{z}$.

Ex. 5
The function $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable on $\mathbb{C}$, with $f(0) = 0$, and $u(x,y) = e^x(x\cos y - y\sin y)$. Find v, and hence show that $f(z) = ze^z$.

Ex. 6
(Exam problem 2002)

(i) If $f(x+iy) = u(x,y) + iv(x,y)$ and $f(z)$ is differentiable at $z = c$, prove that, at c, u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(ii) If $z = x + iy$, write ze^z in the form $u(x,y) + iv(x,y)$.

(iii) Let $f(z)$ be differentiable at z, $f(0) = 0$, and
$f(z) = u + iv$ with $u = x^3 - 3xy^2$. Find f.

(iv) By considering Laplace's equations, or otherwise, show that if
$u(x,y) = x^3 - 2xy^2$ there is no differentiable function $f(z)$ with
$f(x+iy) = u(x,y) + iv(x,y)$.

An electronic version of this problem sheet is available at
http://www.ma.rhul.ac.uk/~elsholtz/04mt290/lecture.html
Recommanded reading:
Murray R. Spiegel: Theory and problems of complex variables. (510.76 Spi).
Donald W. Trim: Introduction to Complex analysis and its applications. (515.24 Tri).
There are copies of these (and a few other books) on the restricted book shelves. Especially the book by Spiegel has numerous solved exercises.