Problem sheet 5
Feb 10th, 2004

MT290 Complex variable

Ex. 1
Let C be the triangular contour joining $0, 1, 1+i$ taken in the anticlockwise direction. Sketch C and write down the equations of each of the three lines making up C. Hence evaluate

$$\int_C \operatorname{Re} z \, dz.$$

Ex. 2
Let γ be the circle with centre 0 and radius r taken anticlockwise. Evaluate (for all r), using the definition of a contour integral,

$$\int_\gamma z^n \, dz,$$

where $n \in \mathbb{Z}$.

Ex. 3
Throughout this question $|z| < 2$. Let

$$f(z) = \frac{1}{z} + g(z)$$

where $|g(z)| < A$ (a fixed constant). Let γ_r be the circle: $re^{it}, \ 0 \le t \le 2\pi$. Use the estimation lemma and question 2 to show that for $r < 2$

$$\left| \int_{\gamma_r} f(z) \, dz - 2\pi i \right| \le 2\pi A r.$$

Deduce that

$$\int_{\gamma_r} f(z) \, dz \longrightarrow 2\pi i$$

as $r \to 0$. [*Note:* there is nothing special about the $|z| < 2$ here, it's there just to restrict our attention to a finite region]

Ex. 4
4. Show that $|e^{iz}| \le 1$ when $\operatorname{Im} z \ge 0$. Let

$$I_R = \int_{\gamma_R} \frac{e^{iz}}{1+z^2} \, dz,$$

where γ_R is the semicircle given by $\phi(t) = Re^{it}, \ 0 \le t \le \pi$. Show that, for all large R,

$$|I_R| \le \frac{2\pi}{R}.$$

Hence deduce that $I_R \to 0$ as $R \to \infty$.

An extra example

(Compare our discussion of $\int_\gamma \frac{1}{z^2+4}\, dz$.)
Let $f(z)$ be a non-constant polynomial of degree at least two, and C(R) a circle centred on 0 with radius R. Use the estimation lemma to show that

$$\lim_{R\to\infty} \int_{C(R)} \frac{1}{f(z)}\ dz \ = \ 0.$$

Hence show, using the deformation of contours theorem, that if all the roots of $f(z) = 0$ lie within the circle $|z| = R$, then

$$\int_{C(R)} \frac{1}{f(z)} dz \ = \ 0.$$

Solution: Since $f(z)$ has degree at least 2 we must have, when $|z| = R$, $|f(z)| > cR^2$ for all large R, where $c > 0$ is any constant. Using the fact that $|f(z)| > (1-\epsilon)|a_k||z|^k$, where a_k is the leading coefficient of f: so we could take $c = \frac{1}{2}|a_k|$). Hence, by the estimation lemma:

$$\left| \int_{C(R)} \frac{1}{f(z)}\ dz \right| \le \frac{\pi R}{cR^2} \to 0 \ \text{ as } R \to \infty.$$

Now, if all the roots of $f(z) = 0$ lie within the circle $|z| = R$, then the function $\frac{1}{f(z)}$ is differentiable on some domain containing both $C(R)$ and a larger circle $C(R')$. Hence

$$\int_{C(R)} \frac{1}{f(z)}\ dz \ = \ \int_{C(R')} \frac{1}{f(z)}\ dz.$$

Since the right hand side tends to zero as R' tends to infinity, we conclude that the integral is zero as required. (If something is both constant and tends to zero, then it must be zero !)