Problem sheet 6
Feb 17th, 2004

MT290 Complex variable

The last problem on this sheet directly corresponds to the lecture from Monday 16th Feb. It may be useful for revision.
With problem 4 you will possibly want to wait for Thursday's lecture.
Problems 1,2, and 4 on sheet 5, problem 3 on sheet 4, and problem 5 on sheet 3 are strongly recommended for revision.

Ex. 1
Evaluate the following integrals around C, the circle $|z| = 1$ taken anticlockwise, using Cauchy's integral formula (or the formula for derivatives if appropriate).

$$\text{(a)} \int_C \frac{\cos z}{z}\, dz; \qquad \text{(b)} \int_C \frac{e^{\pi z}}{(z-\frac{1}{4})^2(z-4)}\, dz.$$

Ex. 2
Solve the quadratic equation $x^2 - 4x + 5 = 0$. Hence write $x^2 - 4x + 5 = (x-\alpha)(x-\beta)$. Use the semicircular contour $\gamma(R)$: $(-R, R) \cup \{Re^{it} : 0 \le t \le \pi\}$ to evaluate the following two integrals:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2 - 4x + 5}\, dx, \qquad \int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 4x + 5}\, dx.$$

Ex. 3
Show that

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2 - 2x + 2}\, dx = -\frac{\pi}{e^{\pi}}, \qquad \int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 - 2x + 2}\, dx = 0.$$

Ex. 4
Factorise the polynomial $6x^2+13x+6$: you should get two factors $(ax+b)(bx+a)$ with a, b both positive integers. Hence evaluate the integral

$$\int_0^{2\pi} \frac{1}{13 + 12\cos\theta}\, d\theta.$$

Ex. 5
(More difficult) Let

$$I_n = \int_0^{2\pi} \cos^{2n}\theta\; d\theta.$$

(Perhaps you saw somewhere Wallis's formulae for integrals like these). Show by complex variable techniques that

$$I_n = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}.$$

Hint. Make the usual substitution to convert the integral to a contour integral around $|z| = 1$. Use the binomial theorem to expand out the integrand. All the terms when integrated give zero except the middle one (note that $\dfrac{(2n)!}{(n!)^2}$ is the coefficient of the middle term in the binomial expansion).

Ex. 6

(i) Evaluate in all details $\int_{\gamma_1} z\,dz$, where γ_1 is the line from z_1 to z_2. and show that $\int_{\gamma_2} z\,dz = 0$, where γ_2 is the closed quadrangle consisting of the 4 points z_1, z_2, z_3, z_4. (Assume for simplicity that the side of the quadrangle do not intersect each other). Now compare with Cauchy's theorem.

(ii) Convince yourself (with a few less details) that you can similarly do $\int_{\gamma_1} z^3\,dz$ and $\int_{\gamma_2} z^3\,dz$.

(iii) $\int_{\gamma_2} \frac{1}{z}\,dz$, where this time the 4 points all lie on the circle $|z| = 2$. Explain the difference between (iii) and (iv).

Discuss the orientation. Discuss (using the the deformation of contours theorem) whether it matters or not, that the points are on the circle.

(iv) Now, assuming Cauchy's theorem and assuming that f satisfies its hypotheses, show that $\int_{\gamma_3} f(z)\,dz = \int_{\gamma_4} f(z)\,dz$, where γ_3 and γ_4 are any paths from z_1 to z_2. Discuss $f(z) = z^3$ and $f(z) = \frac{1}{z}$ as examples.

(v) You know that $\varphi(t) = z = z_1 + (z_2 - z_1)t, 0 \leq t \leq 1$ describes the line from z_1 to z_2. Which curve is described by: $\varphi(t) = z = z_1 + (z_2 - z_1)t^3, 0 \leq t \leq 1$? (Perhaps you will be surprised). Explain your observation. Then evaluate $\int_{\gamma_1} z\,dz$, where γ_1 is given by this new φ, again and compare the result with (i).

If you had any gaps in the minitest, revise those topics.
Revise your knowledge on power (Taylor) series. (MT 172).
Revise your knowledge on the convergence of series.

An electronic version of the problem sheets 1-6 and solutions 1-4 is available at
http://www.ma.rhul.ac.uk/~elsholtz/04mt290/lecture.html
Please tell me if you find any errors. I slightly edited problem 6(ii) on sheet 2 and the solution to sheet 1, problem 7
and problem 5 on sheet 2.