Ex. 1

Revise previous chapters and problem sheets. Do the problems from sheet 6 you haven't done so far (and hand them in!) For problems 2 and 3 on sheet 6 work with e^{iz}.

Ex. 2

$$\text{Let} \qquad I(a,b) = \int_0^{2\pi} \frac{1}{a + b\cos t}\, dt.$$

For which (a,b) do you expect that $I(a,b)$ is a real positive number? Fix $a = 1$ and evaluate $I(1,b)$ for all $0 \le b < 1$. You will want to show that the denominator of the corresponding rational function has precisely one root inside the unit circle.

Ex. 3

The function $f(z) = u(x,y) + iv(x,y)$ is differentiable for all $z = x + iy$, and $f(0) = 0$. Given that $u(x,y) = \sin x \cosh y$, show, using the Cauchy-Riemann equations, that $f(z) = \sin z$. *Hint:* You will need to use results like $\sin(iy) = i \sinh y$ and remember those old trig identities like $\sin(A+B) = \ldots$.

Ex. 4

Let γ be the circle with centre 0 and radius 2, taken anticlockwise. Use Cauchy's integral formula, or the formula for derivatives, to evaluate the following integrals:

$$\int_\gamma \frac{z^2}{z-i}\, dz, \qquad \int_\gamma \frac{\sin \pi z}{(z-1)^2}\, dz.$$

Ex. 5

(This exercise shall help to understand the estimation lemma.)

i) Let (in real analysis) $f(x) = 10 + \sin x$. Sketch the function. Give a lower and an upper bound on $|\int_3^7 f(x)\, dx|$.

ii) Let C be any semicircle Re^{it} ($t_0 \le t \le t_0 + \pi$) around the origin. Give in all details an upper bound on the integral
$$\int_C \frac{z}{z^3}\, dz.$$

iii) Use the estimation lemma to get (in detail) an upper bound on
$$\int_C \frac{2z+1}{z^3+1}\, dz.$$

Ex. 6

(This problem shows that a) not only semicircle contours help to evaluate integrals along the real line; b) that you can reduce an integral like $\int e^{-x^2} \cos ax\, dx$ (here with $a = 1$) to $\int e^{-x^2}, dx$. The latter is not really easy, but at least well known so that you can use it here.)

Let $f : \mathbb{C} \to \mathbb{C}$ be the function with $f(z) = e^{-z^2}$. Let $R(K)$ denote the rectangle defined by the four points $P_1 = -K + 0i, P_2 = +K + 0i, P_3 = +K + \frac{1}{2}i, P_4 = -K + \frac{1}{2}i$.

Let γ_1 denote the path along the edge connecting P_1 and P_2,
Let γ_2 denote the path along the edge connecting P_2 and P_3,
Let γ_3 denote the path along the edge connecting P_3 and P_4,
Let γ_4 denote the path along the edge connecting P_4 and P_1.

i) Draw the integration contour in the Argand diagram.

ii) Show that $\int_{\partial R(K)} f(z)\, dz = 0$.

iii) Show that $\lim_{K\to\infty} \int_{\gamma_2} f(z)\, dz = 0$, and similarly $\lim_{K\to\infty} \int_{\gamma_4} f(z)\, dz = 0$. Use the above results, and (without proof) the well known result $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ to conclude that $\int_0^\infty e^{-x^2} \cos x\, dx = \frac{\sqrt{\pi}}{2e^{1/4}}$.