Problem sheet 8
March 2nd, 2004

MT290 Complex variable

Ex. 1
Give the radius of convergence for each of the following series:
a. $\sum_{n=1}^{\infty} \frac{z^n}{n^3}$; **b.** $\sum_{n=0}^{\infty} \frac{z^n}{5^n}$; **c.** $\sum_{n=0}^{\infty} \frac{z^n}{n^3+5^n}$; **d.** $\sum_{n=0}^{\infty} z^n 5^{-n^2}$

Ex. 2
Find power series for the following functions about the points stated and give the radius of convergence for each of the series.
a. $\frac{1}{2-z}$ about $z=0$; **b.** $\frac{1}{2-z}$ about $z=12$; **c.** $\frac{5}{(1-z)(4+z)}$ about $z=0$.
d. e^z about $z=i$. **e.** $\frac{1}{3-z}$ about $z=4i$.

Ex. 3
Determine the radius of convergence for:

$$a)\ \sum_{n=0}^{\infty} \frac{z^n}{7^n}, \qquad b)\ \sum_{n=0}^{\infty} \frac{z^{5n}}{7^n} \qquad c)\ \sum_{n=0}^{\infty} \frac{z^{bn}}{a^n} \qquad d)\ \sum_{n=0}^{\infty} \frac{z^n}{n^r} \qquad e)\ \sum_{n=0}^{\infty} \frac{z^n}{\binom{3n}{n}}.$$

where in c) a and b and in d) r are positive real constants. Recall that $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ (see also exercise on $n!$ below).

Ex. 4
Determine the power series of $\sin z$ in two different ways:
a) Use the definition of sin in terms of the complex exp function.
b) Use $\frac{d^2 \sin z}{dz^2} = -\sin z,\ \cos(0) = 1$ and the fact that sin is an odd function.
c) Use the definition of $\sinh z$ in terms of the exp to find its power series. Compare with $\sin z$ and deduce that $\sin(iz) = i \sinh z$.

Can you simplify the above calculations by means of Taylor's theorem?

Ex. 5
Stirling's formula is: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Use your pocket calculator to convince yourself for $n = 10, 20, 30, ...$ List some values of $\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$
Use this to find an approximation for $\binom{2n}{n}$ and $\binom{3n}{n}$.
Try to prove a much weaker form of Stirling's formula as follows:
$\ln(n!) = \ln 1 + \ln 2 + \ldots + \ln n$. Approximate the sum by an integral. Find a) a lower and b) an upper bound for $n!$. If these bounds are not too far apart you have a good approximation to $n!$.