Problem sheet 9
March 9th, 2004

# MT290 Complex variable

**Ex. 1**
Remark: Since not everybody appreciated this problem, let's do it again; work through the solution I gave out. You can hand it in.
If you successfully did this problem before, then work out $\int_0^\infty e^{-x^2} \cos(ax)\, dx$ instead.
This problem shows that a) not only semicircle contours help to evaluate integrals along the real line; b) that you can reduce an integral like $\int e^{-x^2} \cos ax\, dx$ to $\int e^{-x^2}, dx$. The latter is not really easy, but at least well known so that you can use it here.
Let $f : \mathbb{C} \to \mathbb{C}$ be the function with $f(z) = e^{-z^2}$. Let $R(K)$ denote the rectangle defined by the four points $P_1 = -K + 0i, P_2 = +K + 0i, P_3 = +K + \frac{1}{2}i, P_4 = -K + \frac{1}{2}i$.
Let $\gamma_1$ denote the path along the edge connecting $P_1$ and $P_2$,
Let $\gamma_2$ denote the path along the edge connecting $P_2$ and $P_3$,
Let $\gamma_3$ denote the path along the edge connecting $P_3$ and $P_4$,
Let $\gamma_4$ denote the path along the edge connecting $P_4$ and $P_1$.

Note: it is advisable to use a parametrisation for the contour lines that keeps $z$ simple but shifts any difficulty to the boundaries. For example, for $\gamma_1$ use $\varphi(t) = z = t$, where $-K \leq t \leq K$. This keeps $e^{-z^2}$ much simpler than $z = -K + 2tK$, with $0 \leq t \leq 1$. So, which simple parametrisation do you get for $\gamma_2$ etc?

i) Draw the integration contour in the Argand diagram.

ii) Show that $\int_{\partial R(K)} f(z)\, dz = 0$. Here $\partial R(K)$ denotes the boundary of the rectangle $R(K)$.

iii) Show that $\lim_{K\to\infty} \displaystyle\int_{\gamma_2} f(z)\, dz = 0$, and similarly $\lim_{K\to\infty} \displaystyle\int_{\gamma_4} f(z)\, dz = 0$. Use the above results, and (without proof) the well known result $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ to conclude that $\int_0^\infty e^{-x^2} \cos x\, dx = \frac{\sqrt{\pi}}{2e^{1/4}}$.

**Ex. 2**
(Use the method of Monday 8th of March.)
Find power series for the following functions about the points stated and give the radius of convergence for each of the series.
**a.** $\dfrac{1}{2-z}$ about $z = 0$; **b.** $\dfrac{1}{2-z}$ about $z = 12$; **c.** $\dfrac{5}{(1-z)(4+z)}$ about $z = 0$.
**d.** $e^z$ about $z = i$. **e.** $\dfrac{1}{3-z}$ about $z = 4i$.

**Ex. 3**
Assuming that it is alright to integrate a power series term by term within its radius of convergence (it is !) use the series for $(1+z)^{-1}$ to obtain the power series:

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n. \qquad (*)$$

What is the radius of convergence of this series? Let $z = iy$ in (*). Take the imaginary part to obtain the series for $\arctan y$.

**Ex. 4**
Find the first few coefficients of the Taylor series of $\tan z$ in two different ways.

a) using $c_n = \frac{f^{(n)}(0)}{n!}$. When you are tired of differentiating try

b)

$$\tan z = \frac{\sin z}{\cos z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} \mp}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} \mp} = c_0 + c_1 z + c_2 z^2 + \dots$$

Multiply by $\cos z$ and find $c_0, c_2, c_4, \dots$ Then find, $c_1$, from this $c_3$ etc.

**Ex. 5**
Complete the following explanation of Taylor's theorem: Perhaps you wondered where this formula $c_n = \frac{f^{(n)}(0)}{n!}$ comes from.
Let's try to filter out the coefficient $c_k$ from $f(z) = \sum_{n=0}^{\infty} c_n z^n$. A good filter is our favourite integral $\int_{|z|=1} z^n\, dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{otherwise.} \end{cases}$
$\int_{|z|=1} f(z)\, dz$ would not lead anywhere, for a differentiable function this is just 0. But let's try

$$\int_{|z|=1} \frac{f(z)}{z^{k+1}}\, dz = \int_{|z|=1} \frac{\sum_{n=0}^{\infty} c_n z^n}{z^{k+1}}\, dz = \int_{|z|=1} \sum_{n=0}^{\infty} c_n \frac{z^n}{z^{k+1}}\, dz$$

Now go on, exchange the integration and summation (you are allowed to do it!), use our filter, combine with Cauchy's integral formulae and find the expression for $c_k$.