Problem sheet 2
Jan 18th 2005

MT290 Complex variable

Ex. 1
Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Define: f is continuous at $c \in \mathbb{C}$.
Use the definition to show that f with $f(z) = z^3$ is continuous everywhere.

Ex. 2
Simplify

$$a) \qquad \left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{201} ; \qquad b) \quad (1+i)^{2n} + (1-i)^{2n}.$$

Ex. 3

a) Draw in the Argand diagram the sequence of points $z^0, z^1, z^2, \ldots$ for $z_1 = i, z_2 = \exp(\frac{2\pi i}{12}), z_3 = 1.1\exp(\frac{2\pi i}{12}), z_4 = 0.9\exp(\frac{2\pi i}{12})$.

b) Given z in modulus argument form, give a general formula for z^n and for $z^{1/n}$.

c) For which complex numbers is $\lim_{n\to\infty} z^n = \infty$, and for which $\lim_{n\to\infty} z^n = 0$. For which complex numbers does the limit not exist?

For the specialists: describe as precise as possible what happens, if $|z| = 1$.

Ex. 4
Sketch the following sets; (for (a)-(c) you will want to write $z = x + iy$):

(a) $\{z : \text{Im } z^2 > 1\}$; (b) $\{z : \text{Re } z^2 \le 1\}$;
(c) $\{z : \text{Re } z^2 > 1, x > 1, y^2 < 4\}$; (d) $\{z : \arg z = \frac{\pi}{6}, 0 < |z| < 1\}$.

Ex. 5
Sketch the following curves:
(a) $\phi(t) = t + i|t-1|, \quad 0 \le t \le 2$
(b) $\phi(t) = 3\sin t + 4i\cos t, \quad 0 \le t \le 2\pi$
(c) $\phi(t) = \sin t + i\sin 2t, \quad 0 \le t \le 2\pi$.

(The following problems use material from the 2nd half of the week).

Ex. 6
Find the derivatives $\frac{df}{dz}$ and show that the Cauchy-Riemann equations are satisfied for the functions $f_1(z) = z^4$ and $f_2(z) = e^{2z}$.

The following two problems are for the specialists (not examinable):

Ex. 7
Here is an alternative construction of the complex numbers.
Let $\mathbb{F} = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x, y, \in \mathbb{R} \right\}$. Show that $\mathbb{F}$, with the usual matrix addition and multiplication, is a field. (I.e. show that addition and multiplication satisfies the axioms of a commutative group; check the distributive law.) What is the mutiplicative identity e?
Find an element f with $f^2 = -e$.
Use this to define a bijective map $\varphi : \mathbb{F} \to \mathbb{C}$ which preserves the whole structure of addition and multiplication; i.e. $a + b = c$ in $\mathbb{F}$ becomes $\varphi(a) + \varphi(b) = \varphi(c)$ in $\mathbb{C}$, etc.

Ex. 8
Now, let $\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : z, w, \in \mathbb{C} \right\}$. Show that $\mathbb{H}$, with the usual matrix addition and multiplication is, well, almost a field. (I.e. show that addition and multiplication satisfies the axioms of a group, but the multiplication is not commutative; check the distributive law.) (In algebra such a structure is called a skew field. The elements of $\mathbb{H}$ are called quaternions, they have many interesting properties...

The problem sheets will be available (probably with a few days delay) at
http://www.ma.rhul.ac.uk/∼elsholtz/WWW/lectures/0405mt290/lecture.html
(The URL given last week was incorrect.)
The workshops have plenty of places left!
Thursday 3pm, Friday 12, and 1pm.