Problem sheet 3
Jan 25th 2005

MT290 Complex variable

Ex. 1
Let $f(x+iy) = x^2 + 2ixy$. Writing $f = u(x,y) + iv(x,y)$ find the partial differentials of u and v. Find where the Cauchy-Riemann equations are satisfied and hence find where this function is differentiable (the answer is a simple line).

Ex. 2
Let $f(x+iy) = y(1-x^2) - iy^2x$. Find the set of points where this function is differentiable and sketch the set on an Argand diagram. Do the real and imaginary parts of this function satisfy Laplace's equation?

Ex. 3
The function $f(x+iy) = u(x,y)+iv(x,y)$ is differentiable on $\mathbb{C}$, with $f(0) = 0$, and $u(x,y) = e^x(x\cos y - y\sin y)$. Find v, and hence show that $f(z) = ze^z$.

Ex. 4
(Exam problem 2002)

(i) If $f(x+iy) = u(x,y) + iv(x,y)$ and $f(z)$ is differentiable at $z = c$, prove that, at c, u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(ii) If $z = x + iy$, write ze^z in the form $u(x,y) + iv(x,y)$. Hence verify that ze^z satisfies the Cauchy-Riemann equations.

(iii) Let $f(z)$ be differentiable at z, $f(0) = 0$, and $f(z) = u + iv$ with $u = x^3 - 3xy^2$. Find f.

(iv) By considering Laplace's equations, or otherwise, show that if $u(x,y) = x^3 - 2xy^2$ there is no differentiable function $f(z)$ with $f(x+iy) = u(x,y) + iv(x,y)$.

(For the following 2 problems you may want to wait till the 2nd half of the week:)

Ex. 5
Determine the principal value of the logarithm of each of the following complex numbers: (a) $-1 - i\sqrt{3}$, (b) $i - \sqrt{3}$, (c) $1 - i$, (d) $\dfrac{i-\sqrt{3}}{i-1}$. *Hint:* Some of the work has already been done on sheet 1 !

Ex. 6
Find the principal value, and all the other values, of the complex powers:
(a) $i^{\frac{2}{\pi}}$, (b) 1^{2i} (c) $(i-\sqrt{3})^i$, (d) $(1-i)^{1+i}$.

There are many more places for the workshops, especially in the two workshops on Fridays. It's your opportunity to get help!