Problem sheet 4
Feb 1st 2005

MT290 Complex variable

Ex. 1
Transform each of the following equations into a quadratic equation in e^z and hence find all the solutions for $z \in \mathbb{C}$ to

(a) $\cosh z = -1$; (b) $\sinh z = \frac{i\sqrt{3}}{2}$.

Ex. 2
Let $f(x+iy) = u(x,y) + iv(x,y)$. Suppose that f is differentiable as a function of a complex variable, and that $v(x,y) = (u(x,y))^2$. Use the Cauchy-Riemann equations to show that

$$\frac{\partial u}{\partial x} = -4u^2 \frac{\partial u}{\partial x}$$

and thus deduce that $f(z)$ is constant. (Note: $1 + 4u^2 \neq 0$!)

Ex. 3
By writing $\tan z$ in terms of $w = e^{iz}$ write down the quadratic equation w must satisfy if

$$\tan z = a.$$

Hence find z if $\tan z = \sqrt{3} - 2i$. By considering the quadratic equation which w satisfies, and using the fact that $e^z = 0$ has no solution, show that $\tan z$ takes all values in $\mathbb{C}$ with two exceptions.

Ex. 4
Sketch the following sets and investigate whether they are *open,* and/or *connected* (for (a)-(c) you will want to write $z = x + iy$):

(a) $\{z : \text{Im } z^2 > 1\}$; (*b*) $\{z : \text{Re } z^2 \leq 1\}$;
(c) $\{z : \text{Re } z^2 > 1, x > 1, y^2 < 4\}$; (*d*) $\{z : \arg z = \frac{\pi}{6}, 0 < |z| < 1\}$.

Ex. 5
Sketch and investigate the following contours to see whether or not they are *smooth, piecewise smooth, simple, closed*: (a) $\phi(t) = t + i|t-1|, \quad 0 \leq t \leq 2$
(b) $\phi(t) = 3\sin t + 4i\cos t, \quad 0 \leq t \leq 2\pi$
(c) $\phi(t) = \sin t + i\sin 2t, \quad 0 \leq t \leq 2\pi$.

For the two problems above compare your earlier sketches on problem sheet 2.

Ex. 6
Let $z = x + iy, x > 0, y > 0$. Express $\log|z|$ and $\arg z$ as functions of x and y (so don't use i: $\log|z| = \log|x+iy|$). Thus show that the function $f(z) = \log|z| + i\arg z$ satisfies the Cauchy-Riemann equations for $x, y > 0$. Since the derivatives are continuous in this region this implies that $f(z)$ is differentiable. Show that $f'(z) = \frac{1}{z}$.