Problem sheet 7
Feb 22nd 2005

MT290 Complex variable

**Ex. 1**
Let $\gamma$ be the circle with centre 0 and radius 2, taken anticlockwise. Use Cauchy's integral formula, or the formula for derivatives, to evaluate the following integrals:

$$\int_\gamma \frac{z^2}{z-i}\,dz, \qquad \int_\gamma \frac{\sin \pi z}{(z-1)^2}\,dz.$$

**Ex. 2**
Show that

$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{x^2-2x+2}\,dx = -\frac{\pi}{e^\pi}, \quad \int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2-2x+2}\,dx = 0.$$

**Ex. 3**
Evaluate the integral

$$\int_0^{2\pi} \frac{1}{13+12\cos\theta}\,d\theta.$$

**Ex. 4**

$$\text{Let} \qquad I(a,b) = \int_0^{2\pi} \frac{1}{a+b\cos t}\,dt.$$

For which $(a,b)$ do you expect that $I(a,b)$ is a real positive number? Fix $a=1$ and evaluate $I(1,b)$ for all $0 \le b < 1$. You will want to show that the denominator of the corresponding rational function has precisely one root inside the unit circle.

**Ex. 5**
(More difficult) Let

$$I_n = \int_0^{2\pi} \cos^{2n}\theta\,d\theta.$$

(Perhaps you saw somewhere Wallis's formulae for integrals like these). Show by complex variable techniques that

$$I_n = \frac{\pi(2n)!}{2^{2n-1}(n!)^2}.$$

*Hint.* Make the usual substitution to convert the integral to a contour integral around $|z|=1$. Use the binomial theorem to expand out the integrand. All the terms when integrated give zero except the middle one (note that $\dfrac{(2n)!}{(n!)^2}$ is the coefficient of the middle term in the binomial expansion).

**Note: Workshops from Feb. 24th onwards:**
**Thursday 11am: 325**
**Thursday 3pm: 325**
**But no Friday workshops any longer.**

Some problems for revision.

**Ex. 6**
The function $f(z) = u(x, y) + iv(x, y)$ is differentiable for all $z = x + iy$, and $f(0) = 0$. Given that $u(x, y) = \sin x \cosh y$, show, using the Cauchy-Riemann equations, that $f(z) = \sin z$. *Hint:* You will need to use results like $\sin(iy) = i \sinh y$ and remember trigonometric identities like $\sin(x + y) = \ldots$.

**Ex. 7**

(i) Evaluate in all details $\int_{\gamma_1} z\,dz$, where $\gamma_1$ is the line from $z_1$ to $z_2$, and show that $\int_{\gamma_2} z\,dz = 0$, where $\gamma_2$ is the closed quadrangle consisting of the 4 points $z_1, z_2, z_3, z_4$. (Assume for simplicity that the side of the quadrangle do not intersect each other). Now compare with Cauchy's theorem.

(ii) Convince yourself (with a few less details) that you can similarly do $\int_{\gamma_1} z^3\,dz$ and $\int_{\gamma_2} z^3\,dz$.

(iii) $\int_{\gamma_2} \frac{1}{z}\,dz$, where this time the 4 points all lie on the circle $|z| = 2$. Discuss the orientation. Discuss (using the the deformation of contours theorem) whether it matters or not, that the points are on the circle.

(iv) Now, assuming Cauchy's theorem and assuming that $f$ satisfies its hypotheses, show that $\int_{\gamma_3} f(z)\,dz = \int_{\gamma_4} f(z)\,dz$, where $\gamma_3$ and $\gamma_4$ are any paths from $z_1$ to $z_2$. Discuss $f(z) = z^3$ and $f(z) = \frac{1}{z}$ as examples.

(v) You know that $\varphi(t) = z = z_1 + (z_2 - z_1)t, 0 \leq t \leq 1$ describes the line from $z_1$ to $z_2$. Which curve is described by: $\varphi(t) = z = z_1 + (z_2 - z_1)t^3, 0 \leq t \leq 1$ ? (Perhaps you will be surprised). Explain your observation. Then evaluate $\int_{\gamma_1} z\,dz$, where $\gamma_1$ is given by this new $\varphi$, again and compare the result with (i).

**Ex. 8**
(This exercise shall help to understand the estimation lemma.)

i) Let (in real analysis) $f(x) = 10 + \sin x$. Sketch the function. Give a lower and an upper bound on $|\int_3^7 f(x)\,dx|$.

ii) Let $C$ be any semicircle $Re^{it}$ $(t_0 \leq t \leq t_0 + \pi)$ around the origin. Give in all details an upper bound on the integral

$$\int_C \frac{z}{z^3}\,dz.$$

iii) Use the estimation lemma to get (in detail) an upper bound on

$$\int_C \frac{2z+1}{z^3+1}\,dz.$$