Problem sheet 8
March 1st 2005

MT290 Complex variable

Ex. 1
Give the radius of convergence for each of the following series:
a. $\sum_{n=1}^{\infty} \frac{z^n}{n^3}$; **b.** $\sum_{n=0}^{\infty} \frac{z^n}{5^n}$; **c.** $\sum_{n=0}^{\infty} \frac{z^n}{n^3+5^n}$; **d.** $\sum_{n=0}^{\infty} z^n 5^{-n^2}$

Ex. 2
Determine the radius of convergence for:

$$a) \sum_{n=0}^{\infty} \frac{z^n}{7^n}, \qquad b) \sum_{n=0}^{\infty} \frac{z^{5n}}{7^n} \qquad c) \sum_{n=0}^{\infty} \frac{z^{bn}}{a^n} \qquad d) \sum_{n=0}^{\infty} \frac{z^n}{n^r} \qquad e) \sum_{n=0}^{\infty} \frac{z^n}{\binom{3n}{n}}.$$

where in c) a and b and in d) r are positive real constants. Recall that $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ and use Stirling's approximation for $n!$ below.

Ex. 3
Determine the integrals, if they exist. Distinguish the various radii R.

$$a) \quad \int_{|z|=R} \frac{dz}{z(z+3)}, \qquad b) \quad \int_{|z|=R} \frac{\sin(\pi z)dz}{z(2z-1)(z-2)}.$$

Ex. 4 (Revision)
The function $f(z) = u(x,y) + iv(x,y)$ is differentiable for all $z = x + iy$, and $f(0) = 0$. Given that $u(x,y) = \sin x \cosh y$, show, using the Cauchy-Riemann equations, that $f(z) = \sin z$. *Hint:* You will need to use results like $\sin(iy) = i \sinh y$ and remember trigonometric identities like $\sin(A+B) = \ldots$.

Ex. 5
Stirling's formula is: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Use your pocket calculator to convince yourself for $n = 10, 20, 30, \ldots$. List some values of $\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$
Use this to find an approximation for $\binom{2n}{n}$ and $\binom{3n}{n}$.
Try to prove a much weaker form of Stirling's formula as follows:
$\ln(n!) = \ln 1 + \ln 2 + \ldots + \ln n$. Approximate the sum by a definite integral like $\int_a^b \ln x\, dx$ with appropriate bounds. Find a) a lower and b) an upper bound for $n!$. If these bounds are not too far apart you have a good approximation to $n!$.

Ex. 6 (More difficult)
(This problem shows that a) not only semicircle contours help to evaluate integrals along the real line; b) that you can reduce an integral like $\int e^{-x^2} \cos ax\, dx$ (here with $a = 1$) to $\int e^{-x^2}, dx$. The latter is not really easy, but at least well known so that you can use it here.)
Let $f : \mathbb{C} \to \mathbb{C}$ be the function with $f(z) = e^{-z^2}$. Let $R(K)$ denote the rectangle defined by the four points $P_1 = -K + 0i, P_2 = +K + 0i, P_3 = +K + \frac{1}{2}i, P_4 = -K + \frac{1}{2}i$.
Let γ_1 denote the path along the edge connecting P_1 and P_2,
Let γ_2 denote the path along the edge connecting P_2 and P_3,
Let γ_3 denote the path along the edge connecting P_3 and P_4,
Let γ_4 denote the path along the edge connecting P_4 and P_1.

i) Draw the integration contour in the Argand diagram.

ii) Show that $\int_{\partial R(K)} f(z)\, dz = 0$. Here $\partial R(K)$ denotes the boundary of the rectangle, taken in anticlockwise direction.

iii) Show that $\lim_{K\to\infty} \int_{\gamma_2} f(z)\, dz = 0$, and similarly $\lim_{K\to\infty} \int_{\gamma_4} f(z)\, dz = 0$. Use the above results, and (without proof) the well known result $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$ to conclude that $\int_0^{\infty} e^{-x^2} \cos x\, dx = \frac{\sqrt{\pi}}{2e^{1/4}}$.