Problem sheet 9
March 8th 2005

MT290 Complex variable

Ex. 1
Find power series for the following functions about the points stated and give the radius of convergence for each of the series.
a. $\dfrac{1}{2-z}$ about $z=0$; **b.** $\dfrac{1}{2-z}$ about $z=12$; **c.** $\dfrac{5}{(1-z)(4+z)}$ about $z=0$. **d.** e^z about $z=i$. **e.** $\dfrac{1}{3-z}$ about $z=4i$.

Ex. 2
Determine the power series of $\sin z$ in two different ways:
a) Use the definition of sin in terms of the complex exp function.
b) Use $\frac{d^2 \sin z}{dz^2} = -\sin z,\ \cos(0)=1$ and the fact that sin is an odd function.
c) Use the definition of $\sinh z$ in terms of the exp to find its power series. Compare with $\sin z$ and deduce that $\sin(iz) = i\sinh z$.

Can you simplify the above calculations by means of Taylor's theorem?

Ex. 3
Assuming that it is alright to integrate a power series term by term within its radius of convergence (it is !) use the series for $(1+z)^{-1}$ to obtain the power series:

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n. \qquad (*)$$

What is the radius of convergence of this series? Let $z = iy$ in (*). Take the imaginary part to obtain the series for $\arctan y$.

Ex. 4
Find the first few coefficients of the Taylor series of $\tan z$ in two different ways.

a) using $c_n = \frac{f^{(n)}(0)}{n!}$. When you are tired of differentiating try

b)

$$\tan z = \frac{\sin z}{\cos z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} \mp}{1 - \frac{z^2}{2!} + \frac{z^4}{4!} \mp} = c_0 + c_1 z + c_2 z^2 + \dots$$

Multiply by $\cos z$ and find $c_0, c_2, c_4, \dots$ Then find, c_1, from this c_3 etc.

Ex. 5
Complete the following explanation of Taylor's theorem: Perhaps you wondered where this formula $c_n = \frac{f^{(n)}(0)}{n!}$ comes from.
Let's try to filter out the coefficient c_k from $f(z) = \sum_{n=0}^{\infty} c_n z^n$. A good filter is our favourite integral $\int_{|z|=1} z^n\,dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise.} \end{cases}$
$\int_{|z|=1} f(z)\,dz$ would not lead anywhere, for a differentiable function this is just 0. But let's try

$$\int_{|z|=1} \frac{f(z)}{z^{k+1}}\,dz = \int_{|z|=1} \frac{\sum_{n=0}^{\infty} c_n z^n}{z^{k+1}}\,dz = \int_{|z|=1} \sum_{n=0}^{\infty} c_n \frac{z^n}{z^{k+1}}\,dz$$

Now go on, exchange the integration and summation (you are allowed to do it!), use our filter, combine with Cauchy's integral formulae and find the expression for c_k.

Note: Reminder/Workshops: Thursday 11am: 325
Thursday 3pm: 325
But no Friday workshops any longer.
If you have questions on the course, home work solutions or feel unhappy about certain topics: (in addition to the workshops, office hour...) I might have time to address these during the last week in the lecture. General questions for this are very welcome! Otherwise I will possibly explain some selected old exam problems.