Exercise Sheet 1

MT4540 Combinatorics

To be returned on Wednesday 13th October 2004

1. Prove that
$$k\binom{n}{k} = n\binom{n-1}{k-1}$$
in two ways:
   (a) By reasoning with subsets.
   (b) Using the formula for a binomial number.

2. Prove that
$$\sum_{k=0}^{n} k\binom{m}{k}\binom{n}{k} = n\binom{m+n-1}{n}.$$
[Hint: Question 1, Lemma 1.1.2 and Lemma 1.1.3 might be useful!]

3. Let p be a prime number.
   (a) Show that $\binom{p}{i}$ is divisible by p for all values of i in the range $1 \leq i \leq p-1$. [Hint: Use the following result. Let r and s be integers such that s divides r. Suppose that p divides r but p does not divide s. Then p divides r/s.]

(b) Does the above result hold when $i = 0$ or $i = p$?

(c) Prove that $(a + b)^p - a^p - b^p$ is divisible by p for all integers a and b.

4. Find the number of ways of arranging the letters A,E,M,O,U,Y in sequence so that the words ME and YOU do not occur.

5. Use first year calculus to prove bounds on $n!$. (Hint: $\ln n! = \ln 1 + \ln 2 + ... + \ln n$, now replace the sum by an integral, and show that the error made by this is rather small: draw a picture of the ln function and compare with the discrete step function of the sum)

   You will get bounds like

$$c_1 \left(\frac{n}{e}\right)^n \leq n! \leq c_2 \left(\frac{n}{e}\right)^{n+1}.$$

   Here c_1, and c_2 are positive constants. Try to find in any book or the internet (without proof) a better approximation to $n!$.

An obvious exercise, for each week is:
Go through your lecture notes (and handouts). If you have any question, or don't understand a proof or so, this is a good exercise! Ask your fellow students, look in a book, and of course, ask me! For example in this week you might like to prove the binomial theorem on your own. Or prove any of the other theorems, without looking at the proofs given in the lecture.