Exercise Sheet 3 MT454 Combinatorics
19th October 2004

**Ex. 1**
Let $P = \{a, b, c, d, e, f, u, v\}$. Draw the Hasse diagram for the poset $(P, \leq)$, where:

$$\begin{array}{l} v < a, v < b, v < c, v < d, v < e, v < f, v < u, \\ a < c, a < d, a < e, a < f, a < u, \\ b < c, b < d, b < e, b < f, b < u, \\ c < d, c < e, c < f, c < u \\ d < e, d < f, d < u \\ e < u, f < u. \end{array}$$

**Ex. 2**
Show that there are 16 different posets on the set $X = \{a, b, c, d\}$. [Here 'different' means not order isomorphic.] Draw Hasse diagrams for them all.

**Ex. 3**
Draw the Hasse diagram of the following posets

a) The poset $\mathcal{D}(16)$ of divisors of 16.

b) The power set $\mathcal{P}(\{1, 2, 3\})$,

c) The poset $P \oplus Y$, where $P = \underline{2}$ and $Y$ is given by the following diagram:

d) The poset $Y \oplus P$, where $P$ and $Y$ are defined as above.

e) The poset $\underline{3} \times \underline{3}$.

**Ex. 4**
Let $n$ be a positive integer. So $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $p_1, p_2, \ldots, p_r$ are distinct prime numbers and where $e_1, e_2, \ldots, e_r$ are positive integers. Prove that

$$\mathcal{D}(n) \cong \underline{(e_1 + 1)} \times \underline{(e_2 + 1)} \times \cdots \times \underline{(e_r + 1)}.$$

**Ex. 5**
Let $n$ be a positive integer. Show that $\mathcal{D}(n) \cong \mathcal{D}(n)^\partial$.

**Ex. 6**
Recall that an antichain is a set $Y$ of incomparable elements in a poset (so $x \leq y$ implies $x = y$ for any $x, y \in Y$). The *width* $w(P)$ of a poset $P$ is the maximal size of an antichain in $P$.

a) Find $w(P)$ for each of the posets $P$ below, and verify in each case that $P$ can be written as the union of $w(P)$ chains.

b) Show that if a finite poset P can be written as the union of $n$ chains, then $n \geq w(P)$.

[Not to be handed in: Let $P$ be a finite poset. A famous theorem — Dilworth's Theorem — says that $w(P)$ is equal to the smallest integer $n$ such that $P$ can be written as the union of $n$ chains. Try proving this yourself...]